Report 1 - Differential Geometry

1 Exercise 1.1.2

Exhibit a couple of topological spaces, by describing M and τ and by checking the three properties mentioned above.

Answer :
1) Let's first take M = $\{1, 2, 3\}$ and
$\tau = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}, \emptyset\}$.

(i) We can see that by definition M and $\emptyset$ are in τ.
(ii) When performing a union between two sets, the resulting set that we are looking at can only increase. By using the union, we can only add elements that are from M to the set we consider. Since τ is composed of all the sets possible over M, the stability per union is guaranteed.
Furthermore, in our case, we cannot exceed more than 3 elements in our set. Then, we can easily see that there is no problem with infinite union.
(iii) The possible intersections are of size 2, 1 or 0. Since τ is composed of all sets over M of size 2, 1 or 0 and since adding another intersection can only make the size decrease, we have the stability over finite intersection.

Now, let's go a little further and consider the case where M = $\{1, 2, 3, 4\}$ and $\tau = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, \emptyset\}$.

We can still guarantee all three properties. Adding an element in M is not an issue.
Let's study in what case we can have an issue.

$\tau = \{\{1\}, \{2\}, \{3\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, \emptyset\}$
Breaks the union rule :
$\{1\} \cup \{2\} = \{1, 2\}$
It is not a topological space.

$\tau = \{\{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, \emptyset\}$
Breaks the intersection rule :
$\{1, 2\} \cap \{1, 3\} = \{1\}$
It is not a topological space.

Now let's study what happens if we want to construct a topological space from two topological space on the same set M. If (M, τ_1) and (M, τ_2) are topological spaces, let's study $(M, \tau_1 \cup \tau_2)$.

By definition, M and $\emptyset$ are in $\tau_1 \cup \tau_2$.

Now, if we take a random union over a family of elements of $\tau_1 \cup \tau_2$:

$$\bigcup_{i=1}^{n} A_i = (\bigcup_{j=1}^{n} A_{1,j}) \cup (\bigcup_{k=1}^{n} A_{2,k})$$

Using the properties of commutativity and associativity of the union we can separate the union between elements of τ_1 and elements of τ_2.

Then, using the property of the stability of the union over τ_1 and elements of τ_2 we can rewrite the precedent union as : $\bigcup_{i=1}^{n} A_i = (E_1) \cup (E_2)$
with E_1 an element of τ_1 and E_2 an element of τ_2.

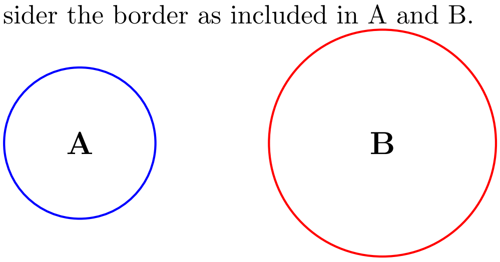
Thus, if we want to construct a new topological space with $\tau_1 \cup \tau_2$ as a base, we need to include every element of the form $(E_1) \cup (E_2)$.

We can do the exact same demonstration with the intersection.

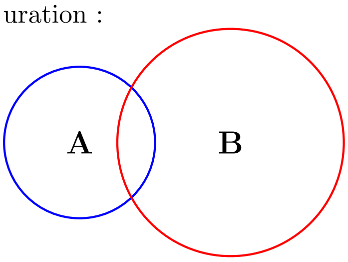
Thus, if we take (M, $\{\{1\}, \emptyset, M\}$) and (M,$\{\{2\}, \emptyset, M\}$), we know that we can construct (M, $\{\{1\}, \{2\}, \{1, 2\}, \emptyset, M\}$).

Now, if we take (M, $\{\{1, 2, 3\}, \emptyset, M\}$), we can construct (M, $\{\{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}, \emptyset, M\}$).

2) Let's now study graphical examples. We do not consider the border as included in A and B.

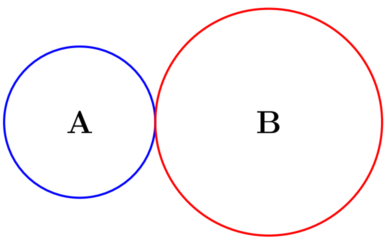


We can have $M{=}A \cup B$ and either $\tau{=}\{A \cup B, \emptyset\}$ (which is always possible) or $\tau{=}\{A \cup B, A, B, \emptyset\}$.
What is interesting to notice is that if we use this configuration :



we cannot use the second τ to define a topological space anymore since $A \cap B$ would not be in M.

Futhermore, we have to be careful when considering the frontier: do we consider the ball as an open ball or not ?



If we consider the frontiers of the balls, the intersection is not stable inside M. If we do not consider it, we do not have a problem. This is linked to the neighborhood

property. If you take an element of the border, you cannot define a neighborhood around it anymore.
3) Finally, let's try to see a more abstract example.
Let M=]0, 10[and τ be all the open segments with the smallest bound being 0 (]0; b[with $b > 0$) and the empty set.

(i) By definition M and the empty set are in τ.
(ii) Let's prove by induction that this topological space guarantees stability by union.

We define the property $\mathcal{P}_n$ as "the union of n open segments on]0; 10[is an open segment.

Base case
Let's take a union of two elements of τ as base case.
$]0, a[\cup]0, b[=]0, max(b, a)[$ with $max(a, b) > 0$. Thus, $]a, b[\cup]c, d[$ is an element of τ

Induction Let's prove that $\mathcal{P}_n$ implies $\mathcal{P}_{n+1}$:

According to $\mathcal{P}_n$, if we take a family on n open segments on M, we can find a_n such that $]0, a_n[$ is equal to their union :

$\bigcup_{i=1}^{n} A_i =]0, a_n[$ with $a_n \in]1; 10[$ and (A_i) a family of open segments of τ.

Now let's see what happens if we do the union of another segment with this family.

$]0, a_n[\cup]0, b[=]0; max(a_n, b)[=]0; a_{n+1}[$ with $]0; a_{n+1}[$ an open segment included in M.

Then $\mathcal{P}_n \Rightarrow \mathcal{P}_{n+1}$ and we have the second property.

iii) For the third property, we can proceed as previously.
We then have defined a topological space.

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