Solution to Homework 7 (4+2+2+5+4=17pts)

Exercise 1.

The logarithm function ln is defined as the inverse of the natural exponential function exp.

(i) We observe that $\exp(\ln(x)) = x$ for $x > 0$. As such, the following equivalences hold:

$$\begin{aligned} \frac{d}{dx}[\exp(\ln(x))] &= \frac{d}{dx}[x] \\ e^{\ln(x)}\frac{d}{dx}[\ln(x)] &= 1 \\ \frac{d}{dx}[\ln(x)] &= \frac{1}{e^{\ln(x)}} = \underline{\underline{\frac{1}{x}}}. \end{aligned}$$

(ii) Let $a = e^x$ and $b = e^y$. As $a, b > 0$ we can write

$$\begin{aligned} \ln(ab) &= \ln(e^x e^y) \\ &= \ln(e^{x+y}) \\ &= x + y \\ &= \ln(e^x) + \ln(e^y) \\ &= \underline{\underline{\ln(a) + \ln(b)}}. \end{aligned}$$

(iii) For $n \in \mathbb{N}$ by the result of part *(ii)* one has

$$\ln(x^n) = \ln(x) + \cdots + \ln(x) = \underline{\underline{n\ln(x)}}.$$

Then, we set $x := z^{\frac{1}{m}}$. As such, by the previous result,

$$\begin{aligned} \ln(z) &= \ln(x^m) \\ \ln(z) &= m\ln(x) \\ \ln(x^{\frac{1}{m}}) &= \underline{\underline{\frac{1}{m}\ln(x)}}. \end{aligned}$$

As a consequence of the previous step and part *(ii)* we have that

$$\begin{aligned} \ln(x^{\frac{n}{m}}) &= \ln\left((x^{\frac{1}{m}})^n\right) \\ &= n\ln(x^{\frac{1}{m}}) \\ &= \underline{\underline{\frac{n}{m}\ln(x)}}. \end{aligned}$$

Following the previous result (and also of part *(ii)*) we have that

$$\begin{aligned} \ln(1) &= \ln\left(x^{\frac{n}{m}}x^{-\frac{n}{m}}\right) \\ 0 &= \ln\left(x^{\frac{n}{m}}\right) + \ln\left(x^{-\frac{n}{m}}\right) \\ \ln\left(y^{-\frac{n}{m}}\right) &= \underline{\underline{-\frac{n}{m}\ln(x)}}. \end{aligned}$$

And thus we have proven that for all $q \in \mathbb{Q}$

$$\ln(x^q) = \underline{\underline{q\ln(x)}}.$$

Exercise 2.

The exponential function, $\exp : \mathbb{R} \to (0, \infty)$ is the function defined by

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} .$$

This function satisfies $\exp' = \exp$, $\exp(0) = 1$, and exp is strictly increasing over $\mathbb{R}$.
Its inverse is the logarithm function $\ln : (0, \infty) \to \mathbb{R}$. For $x > 0$ and $\alpha \in \mathbb{R}$ we set

$$x^\alpha \;:=\; \exp(\, \alpha \ln(x)\,) .$$

One also defines $e := \exp(1) \;=\; 2.718....$ Then one defines

$$\begin{aligned} \exp(\alpha) \;&=\; \exp(\alpha \cdot 1) \\ &=\; \exp(\alpha \cdot \ln(\exp(1))) \\ &=\; \exp(\alpha \ln(e)) \;=\; \underline{\underline{e^\alpha}}. \end{aligned}$$

This justifies the notation $\exp(\alpha) = e^\alpha$.

Exercise 3.

The derivative of $f : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $f(x) = a^x$ $(a > 0)$ is to be calculated.

$$\begin{aligned} f'(x) \;&=\; \frac{d}{dx}\left[e^{x\ln(a)}\right] \\ &=\; e^{x\ln(a)} \cdot \frac{d}{dx}\left[x\ln(a)\right] \\ &=\; \underline{\underline{a^x \ln(a)}}. \end{aligned}$$

The derivative of $g : \mathbb{R}_+^* \to \mathbb{R}$ defined by $g(x) = x^x$ is to be calculated.

$$\begin{aligned} g'(x) \;&=\; \frac{d}{dx}\left[e^{x\ln(x)}\right] \\ &=\; e^{x\ln(x)} \cdot \frac{d}{dx}\left[x\ln(x)\right] \\ &=\; x^x \cdot \left(x \cdot \frac{1}{x} + \ln(x)\right) \\ &=\; \underline{\underline{(1 + \ln(x))x^x}}. \end{aligned}$$

Exercise 4.

a) The limit $\lim_{x \searrow 0} x \ln(x)$ is to be evaluated.

Set $y := -\ln(x)$, so $x = e^{-y}$ and as $x \to 0_+$, $y \to +\infty$. Thus, we rewrite the limit

$$\lim_{x \searrow 0} x \ln(x) \;=\; \lim_{y \to +\infty} -\frac{y}{e^y} = -\lim_{y \to +\infty} \frac{y}{e^y}$$

Since $e^y = 1 + y + \frac{1}{2}y^2 + \cdots > \frac{1}{2}y^2$, one has that

$$\lim_{y \to +\infty} \frac{y}{e^y} < \lim_{y \to +\infty} \frac{y}{\frac{1}{2}y^2} = \lim_{y \to \infty} \frac{2}{y} = 0$$

Therefore $\lim_{x \searrow 0} x \ln(x) = -\lim_{y \to +\infty} \frac{y}{e^y} = \underline{\underline{0\,.}}$

b) The limit $\lim_{x\searrow 0} x^x$ is to be evaluated.

By the result in part *a)* and **Exercise 4.10.**, one has

$$\lim_{x\searrow 0} x^x = \lim_{x\searrow 0} e^{x\ln(x)} = e^{\lim_{x\searrow 0} x\ln(x)} = e^0 = \underline{\underline{1\,.}}$$

c) The limit $\lim_{x\to+\infty} \frac{\ln(x)}{x}$ to be evaluated.

Set $y := \ln(x)$. As a consequence, $x = e^y$. Thus, the limit can be rewritten:

$$\lim_{x\to+\infty} \frac{\ln(x)}{x} = \lim_{y\to+\infty} \frac{y}{e^y} = \underline{\underline{0\,.}}$$

d) The limit $\lim_{x\to+\infty} x^{\frac{1}{x}}$ is to be evaluated.

By the result in part *c)* and **Exercise 4.10.**, one has

$$\lim_{x\to+\infty} x^{\frac{1}{x}} = \lim_{x\to+\infty} e^{\frac{\ln(x)}{x}} = e^{\lim_{x\to+\infty} \frac{\ln(x)}{x}} = e^0 = \underline{\underline{1\,.}}$$

e) For $r > 0$, the limit $\lim_{x\searrow 0} x^r \ln(x)$ is to be evaluated.

Set $y := -\ln(x)$, so $x = e^{-y}$ and as $x \to 0_+$, $y \to +\infty$. Thus, we rewrite the limit

$$\lim_{x\searrow 0} x^r \ln(x) = \lim_{y\to+\infty} -\frac{y}{e^{ry}} = -\lim_{y\to+\infty} \frac{y}{e^{ry}}$$

Since $e^{ry} = 1 + ry + \frac{1}{2}(ry)^2 + \cdots > \frac{1}{2}(ry)^2$, one has that

$$\lim_{y\to+\infty} \frac{y}{e^{ry}} < \lim_{y\to+\infty} \frac{y}{\frac{1}{2}r^2y^2} = \lim_{y\to+\infty} \frac{2}{r^2}\frac{1}{y} = 0\,.$$

Therefore $\lim_{x\searrow 0} x^r \ln(x) = -\lim_{y\to+\infty} \frac{y}{e^{ry}} = \underline{\underline{0\,.}}$

Exercise 5.

a) The limit $\lim_{x\to 0} \frac{\ln(1+x)}{x}$ is to be evaluated.

As $\lim_{x\to 0} \ln(1+x) = 0 = \lim_{x\to 0} x$ and $\frac{d}{dx}[x] \neq 0$, so by applying L'Hôpital's Rule,

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = \frac{1}{1} = \underline{\underline{1\,.}}$$

b) The limit $\lim_{x\to 0}(1+x)^{\frac{1}{x}}$ is to be evaluated.

By the results of part *a)* and **Exercise 4.10.**, one has

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}\ln(1+x)} = e^{\lim_{x\to 0}\frac{\ln(1+x)}{x}} = e^1 = \underline{\underline{e\,.}}$$

c) The limit $\lim_{x\to+\infty} \left(1 + \frac{1}{x}\right)^x$ is to be evaluated.

We set $y := \frac{1}{x}$. As $x \to +\infty$, $y \to 0_+$. Thus, by the result of part *b)* we have

$$\lim_{x\to+\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{y\to 0_+} (1+y)^{\frac{1}{y}} = \underline{\underline{e\,.}}$$

d) The limit $\lim_{x\to+\infty} \left(1 + \frac{r}{x}\right)^x$ is to be evaluated for $r > 0$.

We set $y := \frac{r}{x}$. As $x \to +\infty$, $y \to 0_+$. Thus, by the result of part *b)* we have

$$\lim_{x\to+\infty} \left(1 + \frac{r}{x}\right)^x = \lim_{y\to 0_+} (1+y)^{\frac{r}{y}} = \underline{\underline{e^r\,.}}$$