
Homework 7

Exercise 1 Prove the following properties of the function \ln :

(i) $\ln(x)' = \frac{1}{x}$ for any $x \in (0, \infty)$,

(ii) $\ln(xy) = \ln(x) + \ln(y)$ for any $x, y \in (0, \infty)$,

(iii) $\ln(x^q) = q \ln(x)$ for any $x \in (0, \infty)$ and $q \in \mathbb{Q}$.

Exercise 2 Let us set $e := e^1 = 2.718\dots$. Check that $\ln(e) = 1$ and that $e^x = e^x$.

Exercise 3 Compute the derivative of the following functions:

$$f : \mathbb{R} \ni x \mapsto a^x \in \mathbb{R} \text{ for any } a > 0, \quad g : \mathbb{R}_+^* \ni x \mapsto x^x \in \mathbb{R}.$$

Exercise 4 Compute the following limits:

$$a) \lim_{x \rightarrow 0_+} x \ln(x), \quad b) \lim_{x \rightarrow 0_+} x^x, \quad c) \lim_{x \rightarrow +\infty} \frac{\ln(x)}{x}, \quad d) \lim_{x \rightarrow +\infty} x^{1/x}.$$

What can you say for $\lim_{x \rightarrow 0_+} x^r \ln(x)$ for any $r > 0$?

Exercise 5 Compute the following limits:

$$a) \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x}, \quad b) \lim_{x \rightarrow 0} (1+x)^{1/x}, \quad c) \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x, \quad d) \lim_{x \rightarrow \infty} \left(1 + \frac{r}{x}\right)^x \text{ for any } r > 0.$$