
Solution to Homework 6 (4+3+2+2+2=13pts)

Exercise 1.

Recall that $e^{-x} := \frac{1}{e^x}$. The derivative of $\sinh(x)$ is to be calculated.

$$\begin{aligned}\frac{d}{dx}[\sinh(x)] &= \frac{1}{2}\frac{d}{dx}\left[e^x - e^{-x}\right] \\ &= \frac{1}{2}\left(e^x - (-1)e^{-x}\right) \\ &= \frac{1}{2}\left(e^x + e^{-x}\right) = \underline{\underline{\cosh(x)}}.\end{aligned}$$

The derivative of $\cosh(x)$ is to be calculated.

$$\begin{aligned}\frac{d}{dx}[\cosh(x)] &= \frac{1}{2}\frac{d}{dx}\left[e^x + e^{-x}\right] \\ &= \frac{1}{2}\left(e^x + (-1)e^{-x}\right) \\ &= \frac{1}{2}\left(e^x - e^{-x}\right) = \underline{\underline{\sinh(x)}}.\end{aligned}$$

One observes that for $\sinh(x)$,

i) $\sinh(0) = \frac{1}{2}(e^0 - e^0) = 0$.

ii) As $x \to \infty$, $e^x \to \infty$ and $e^{-x} \to 0$ (vice versa for $x \to -\infty$), so $\lim_{x\to\pm\infty}\sinh(x) \to \pm\infty$.

iii) $\sinh(-x) = \frac{1}{2}(e^{-x} - e^{-(-x)}) = -\frac{1}{2}(e^x - e^{-x}) = -\sinh(x)$, so $\sinh(x)$ is an odd function.

One observes that for $\cosh(x)$,

i) $\cosh(0) = \frac{1}{2}(e^0 + e^0) = \frac{1}{2}(2) = 1$.

ii) As for all x, $e^x > 0$ and $e^{-x} > 0$, then $\cosh(x) > 0$.

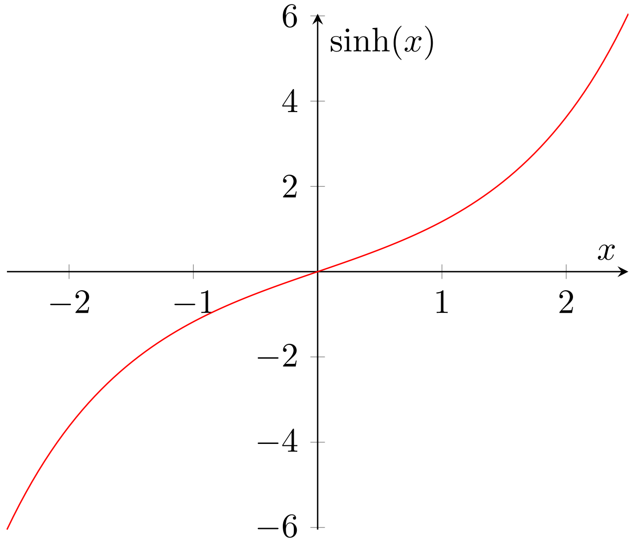
iii) As $x \to \infty$, $e^x \to \infty$ and $e^{-x} \to 0$ (vice versa for $x \to -\infty$), so $\lim_{x\to\pm\infty}\cosh(x) \to \infty$.

iv) $\cosh(-x) = \frac{1}{2}(e^{-x} + e^{-(-x)}) = \frac{1}{2}(e^x + e^{-x}) = \cosh(x)$, so $\cosh(x)$ is an even function.

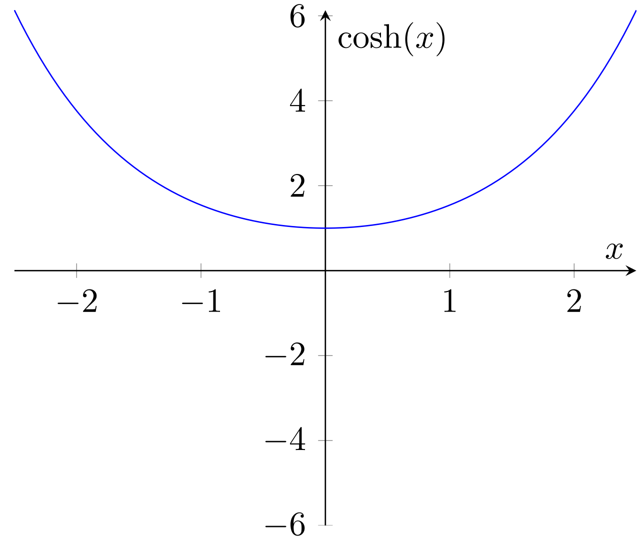
The identity $\cosh^2(x) - \sinh^2(x) = 1$ is to be proven for all x.

$$\begin{aligned}\cosh^2(x) - \sinh^2(x) &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} + e^{-2x} + 2e^x e^{-x}}{4} - \frac{e^{2x} + e^{-2x} - 2e^x e^{-x}}{4} \\ &= \frac{e^{2x} + e^{-2x}}{4} + \frac{1}{2} - \frac{e^{2x} + e^{-2x}}{4} + \frac{1}{2} \\ &= \underline{\underline{1}}.\end{aligned}$$

The graphs of the two functions are drawn below.



(a) The graph of sinh



(b) The graph of cosh

Exercise 2.

Let point P be a point on the curve $y^2 = 4x$. The distance between P and $(2,3)$ is given by

$$s(y) = \sqrt{\left(\frac{y^2}{4} - 2\right)^2 + (y-3)^2} = \sqrt{13 - 6y + \frac{1}{16}y^4}$$

Observe that since $(2,3)$ does not satisfy $y^2 = 4x$, $s(y) > 0$.
To find the minimum value of $s(y)$ we find the critical point. First, calculate $s'(y)$:

$$\begin{aligned} s'(y) &= \frac{1}{2}\left(13 - 6y + \frac{1}{16}y^4\right)^{-\frac{1}{2}} \cdot \frac{d}{dx}\left[13 - 6y + \frac{1}{16}y^4\right] \\ &= -\frac{\frac{1}{4}y^3 - 6}{2\sqrt{13 - 6y + \frac{1}{16}y^4}} = -\frac{\frac{1}{4}y^3 - 6}{s(y)} \end{aligned}$$

And equating $s'(y) = 0$ (and as $s(y) > 0$) we get:

$$\begin{aligned} 0 &= \frac{1}{4}y^3 - 6 \\ 6 &= \frac{1}{4}y^3 \\ y &= 2\sqrt[3]{3}. \end{aligned}$$

And thus the value of x is:

$$\begin{aligned} x &= \frac{1}{4}\left(2\sqrt[3]{3}\right)^2 \\ &= \frac{1}{4} \cdot 4 \cdot \sqrt[3]{9} \\ &= 3^{\frac{2}{3}}. \end{aligned}$$

Therefore the closest point at the curve $y^2 = 4x$ to point (2,3) is $(\sqrt[3]{9}, 2\sqrt[3]{3})$.

To prove the nature of the point, we can use the 2nd derivative test. By derivation,

$$s''(y) = \frac{\frac{3}{2}y^2 s(y) - s'(y)(\frac{1}{2}y^3 - 12)}{4\left(s(y)\right)^2}$$

At $y = 2\sqrt[3]{3}$, $s''(y) > 0$. As such, $(\sqrt[3]{9}, 2\sqrt[3]{3})$ is a minimum point.

Exercise 3.

The required statement is the equivalent to proving that $f(x) = x - \sin(x)$ satisfies $f(x) \geq 0$ for all $x \geq 0$.
First, $f(0) = 0 - \sin(0) = 0 \geq 0$. Then, we calculate $f'(x)$:

$$\frac{d}{dx}\left[x - \sin(x)\right] = 1 - \cos(x)$$

Which means that $\frac{d}{dx}\left[x - \sin(x)\right] \geq 0$, which means that f is an increasing function on $[0, \infty)$.
This proves that $\underline{\underline{f(x) \geq 0 \text{ for all } x \geq 0.}}$

Exercise 4.

Let $(x_0, f(x_0)) = (x_0, (x_0+1)^2)$ be the point of tangency. By differentiation

$$f'(x_0) = \frac{d}{dx_0}\left[(x_0+1)^2\right] = 2x_0 + 2$$

The function whose equation is the line passing through (0,0) and of slope $2x_0 + 2$ is

$$l : \mathbb{R} \ni x \mapsto (2x_0 + 2)x \in \mathbb{R}$$

If this line is also to pass through the point of tangency, then $l(x_0) = f(x_0)$. As such,

$$\begin{aligned}(2x_0 + 2)x_0 &= (x_0 + 1)^2 \\ 2x_0^2 + 2x_0 &= x_0^2 + 2x_0 + 1 \\ x_0^2 - 1 &= 0 \\ (x_0 - 1)(x_0 + 1) &= 0.\end{aligned}$$

Thus one infers that the equations of the two lines are

$$l_1(x) = \underline{\underline{0}} \quad \text{or} \quad l_2(x) = \underline{\underline{4x}}\ .$$

Exercise 5. By the Intermediate Value Theorem, for any $y \in [\alpha, \beta]$ there exists $x \in [a, b]$ such that $y = f(x)$.
We define a function $f^{-1} : [\alpha, \beta] \to [a, b]$ defined by $f^{-1}(y) = x$.
Next, one must show that f^{-1} is uniquely defined. ($\forall y \in [\alpha, \beta]\ \exists$ only one $x \in [a, b]$ such that $y = f(x)$)

To prove that f^{-1} uniquely defined, by contraposition, let us assume that

$$\exists x,\, x' \in [a, b]\,,\, x \neq x' \text{ such that } f(x) = f(x')$$

We have that f is continuous and strictly increasing. As such,

$$x > x' \implies f(x) > f(x') \quad \text{and} \quad x < x' \implies f(x) < f(x')$$

Which proves that the inverse statement (f^{-1} not uniquely defined) is false.

Which proves the statement that $\underline{\underline{f^{-1} \text{ is uniquely defined.}}}$