Solution to Homework 5 (3+4+3+3+4=17pts)

Exercise 1.

Assume that locally, y can be obtained as a function of x satisfying

$$x^2 - y^2 + 3xy + 12 = 0$$

Then differentiating both sides gives

$$\begin{aligned} 0 &= \frac{d}{dx}\left[x^2\right] - \frac{d}{dx}\left[y^2\right] + 3x\frac{d}{dx}[y] + 3y\frac{d}{dx}[x] + 0 \\ 0 &= 2x - 2y\frac{dy}{dx} + 3x\frac{dy}{dx} + 3y \\ (2y - 3x)\frac{dy}{dx} &= 2x + 3y \\ \frac{dy}{dx} &= \frac{2x+3y}{2y-3x} \qquad \text{Whenever } 2y - 3x \neq 0 \end{aligned}$$

Observe that

$$F(-4,2) = (-4)^2 - 2^2 + 3(-4)(2) + 12 = 16 - 4 - 24 + 12 = 0$$

Which means point $(-4,2)$ is in the curve. Then,

$$\frac{dy}{dx}(-4,2) = \frac{2(-4)+3(2)}{2(2)-3(-4)} = \frac{-2}{16} = -\frac{1}{8}$$

A line in $\mathbb{R}^2$ generally can be written as $y = mx + b$. Substituting known values at point $(-4,2)$ gives

$$\begin{aligned} 2 &= \frac{-1}{8}\cdot(-4) + b \\ b &= 2 - \frac{1}{2} = \frac{3}{2} \end{aligned}$$

Therefore the equation of the tangent line to the curve at $(-4,2)$ is

$$y = -\frac{1}{8}x + \frac{3}{2}.$$

Exercise 2.

1) There are two methods about this; one calculation-heavy one and one using the composition rule.

   (a) The Calculation-Heavy Method (longer):
   Let us set

$$a := (x+h)^{\frac{1}{n}} \qquad \text{and} \qquad b := x^{\frac{1}{n}}$$

   Evaluate the expression $(x+h)^{\frac{1}{n}} - x^{\frac{1}{n}} = a - b$:

$$\begin{aligned} (x+h)^{\frac{1}{n}} - x^{\frac{1}{n}} &= (a-b)\cdot\frac{\sum_{k=0}^{n-1} a^{n-k-1}b^k}{\sum_{k=0}^{n-1} a^{n-k-1}b^k} \\ &= \frac{a^n - b^n}{\sum_{k=0}^{n-1} a^{n-k-1}b^k} \\ &= \frac{x+h-x}{\sum_{k=0}^{n-1} a^{n-k-1}b^k} = \frac{h}{\sum_{k=0}^{n-1} a^{n-k-1}b^k} \end{aligned}$$

Then by the definition of the derivative,

$$\begin{aligned}
p'_{\frac{1}{n}}(x) &= \lim_{h\to 0} \frac{(x+h)^{\frac{1}{n}} - x^{\frac{1}{n}}}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{h}{\sum_{k=0}^{n-1} a^{n-k-1} b^k} \\
&= \lim_{h\to 0} \frac{1}{\sum_{k=0}^{n-1} x^{\frac{n-k-1}{n}} x^{\frac{k}{n}}} \\
&= \frac{1}{\sum_{k=0}^{n-1} x^{\frac{n-k-1}{n}} x^{\frac{k}{n}}} \\
&= \frac{1}{\sum_{k=0}^{n-1} x^{\frac{n-1}{n}}} = \frac{1}{\sum_{k=0}^{n-1} x^{1-\frac{1}{n}}} \\
&= \frac{1}{nx^{1-\frac{1}{n}}} = \underline{\underline{\frac{1}{n}\, x^{\frac{1}{n}-1}}}\,.
\end{aligned}$$

(b) The Composition Rule Method (shorter):

Observe that $\forall x,\ x = \left(x^{\frac{1}{n}}\right)^n$. Thus,

$$\begin{aligned}
\frac{d}{dx}[x] &= \frac{d}{dx}\left[\left(x^{\frac{1}{n}}\right)^n\right] \\
1 &= n \cdot \left(x^{\frac{1}{n}}\right)^{n-1} \cdot \frac{d}{dx}\left[x^{\frac{1}{n}}\right] \\
\frac{d}{dx}\left[x^{\frac{1}{n}}\right] &= \frac{1}{n} \cdot \frac{1}{x^{\frac{n-1}{n}}} \\
&= \frac{1}{n}\, x^{\frac{-n+1}{n}} \\
&= \underline{\underline{\frac{1}{n}\, x^{\frac{1}{n}-1}}}\,.
\end{aligned}$$

2) By the result of part 1) we obtain

$$\begin{aligned}
\frac{d}{dx}\left[x^{\frac{m}{n}}\right] &= \frac{d}{dx}\left[\left(x^{\frac{1}{n}}\right)^m\right] \\
&= m \cdot \left(x^{\frac{1}{n}}\right)^{m-1} \frac{d}{dx}\left[x^{\frac{1}{n}}\right] \\
&= m \cdot x^{\frac{m-1}{n}} \cdot \frac{1}{n} x^{\frac{1-n}{n}} \\
&= \frac{m}{n}\, x^{\frac{m-1+1-n}{n}} \\
&= \underline{\underline{\frac{m}{n}\, x^{\frac{m}{n}-1}}}\,.
\end{aligned}$$

3) By the result of part 2) and the result that $\frac{d}{dx}\left[x^{-1}\right] = -x^{-2}$, we find $\frac{d}{dx}\left[x^{-q}\right]$: $(q := \frac{m}{n})$

$$\begin{aligned}
\frac{d}{dx}\left[x^{-q}\right] &= \frac{d}{dx}\left[x^{-\frac{m}{n}}\right] \\
&= -\left(x^{\frac{m}{n}}\right)^{-2} \frac{d}{dx}\left[x^{\frac{m}{n}}\right] \\
&= -x^{-\frac{2m}{n}} \cdot \frac{m}{n} x^{\frac{m}{n}-1} \\
&= -\frac{m}{n} x^{-\frac{m}{n}-1} \\
&= \underline{\underline{-q\, x^{-q-1}}}.
\end{aligned}$$

By putting all of the results together, we conclude that for all $q \in \mathbb{Q}$,

$$\frac{d}{dx}\left[x^q\right] = qx^{q-1}\,.$$

Exercise 3.

a) The derivative of the function defined by $f(x) = \sin\left((2x^2-3)^2\right)$ is to be calculated by composition rule.

$$\begin{aligned} f'(x) &= \cos\left((2x^2-3)^2\right)\frac{d}{dx}\left[(2x^2-3)^2\right] \\ &= \cos\left((2x^2-3)^2\right)\cdot 2(2x^2-3)\cdot\frac{d}{dx}\left[2x^2-3\right] \\ &= \cos\left((2x^2-3)^2\right)\cdot(4x^2-6)\cdot 4x \\ &= (16x^3-24x)\cos\left((2x^2-3)^2\right). \end{aligned}$$

b) The derivative of the function defined by $f(x) = \frac{(x+3)^3}{(2x-3)^2+1}$ is to be calculated by composition rule.

$$\begin{aligned} f'(x) &= \frac{\frac{d}{dx}\left[(x+3)^3\right]\cdot((2x-3)^2+1) - \frac{d}{dx}\left[(2x-3)^2+1\right]\cdot(x+3)^3}{((2x-3)^2+1)^2} \\ &= \frac{3(x+3)^2\cdot\frac{d}{dx}[x+3]\cdot((2x-3)^2+1) - 2\cdot(2x-3)\cdot\frac{d}{dx}[2x-3]\cdot(x+3)^3}{((2x-3)^2+1)^2} \\ &= \frac{3(x+3)^2\cdot((2x-3)^2+1) - 2\cdot(2x-3)\cdot 2\cdot(x+3)^3}{(4x^2-12x+10)^2} \\ &= \frac{(x+3)^2\cdot\left(3(4x^2-12x+10)-4(2x-3)(x+3)\right)}{4(2x^2-6x+5)^2} \\ &= \frac{2(x+3)^2\left(2x^2-24x+33\right)}{4(2x^2-6x+5)^2} = \frac{(x+3)^2\left(2x^2-24x+33\right)}{2(2x^2-6x+5)^2}\,. \end{aligned}$$

c) The derivative of the function defined by $f(x) = \frac{1}{\sin^2(3x)+1}$ is to be calculated by composition rule.

$$\begin{aligned} f'(x) &= -\frac{1}{(\sin^2(3x)+1)^2}\cdot\frac{d}{dx}\left[\sin^2(3x)+1\right] \\ &= -\frac{1}{(\sin^2(3x)+1)^2}\cdot 2\sin(3x)\cdot\frac{d}{dx}\left[\sin(3x)\right] \\ &= -\frac{1}{(\sin^2(3x)+1)^2}\cdot 2\sin(3x)\cdot 3\cos(3x) \\ &= -\frac{6\sin(3x)\cos(3x)}{(\sin^2(3x)+1)^2}\,. \end{aligned}$$

Exercise 4.

L'Hôpital's Rule states that ($x_0 \in I \subset \mathbb{R}$)

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$$

if and only if the following conditions are met:

(a) The functions f and g are differentiable over an open interval $I \subset \mathbb{R}$,

(b) $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$

(c) For all $x \in I$, $x \neq x_0$, $g'(x) \neq 0$.

(d) The limit $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists.

As such, it is imperative to prove that the conditions are met before applying L'Hôpital's Rule.

(*i*) The limit $\lim_{x\to 0} \frac{x-\sin(x)}{x^3}$ is to be evaluated.
By differentiation we obtain

$$\frac{d}{dx}\left[x-\sin(x)\right] = 1-\cos(x) \qquad \frac{d^2}{dx^2}\left[x-\sin(x)\right] = \sin(x) \qquad \frac{d^3}{dx^3}\left[x-\sin(x)\right] = \cos(x)$$

$$\frac{d}{dx}\left[x^3\right] = 3x^2 \qquad \frac{d^2}{dx^2}\left[x^3\right] = 6x \qquad \frac{d^3}{dx^3}\left[x^3\right] = 6$$

Observe that:

(a) The first, second, and third order derivatives of $x-\sin(x)$ and x^3 are all differentiable over $\mathbb{R}$.

(b) $\lim_{x\to 0} x-\sin(x) = \lim_{x\to 0} 1-\cos(x) = \lim_{x\to 0} \sin(x) = \lim_{x\to 0} x^3 = \lim_{x\to 0} 3x^2 = \lim_{x\to 0} 6x = 0.$

(c) $3x^2 \neq 0$ and $6x \neq 0$ for all $x \neq 0$. (while $6 \neq 0$ always)

Which shows that we meet the conditions for L'Hôpital's Rule (last condition to be proven below).
Thus, applying L'Hôpital's Rule 3 times,

$$\begin{aligned}\lim_{x\to 0} \frac{x-\sin(x)}{x^3} &= \lim_{x\to 0} \frac{1-\cos(x)}{3x^2} \\ &= \lim_{x\to 0} \frac{\sin(x)}{6x} \\ &= \lim_{x\to 0} \frac{\cos(x)}{6} \\ &= \underline{\underline{\frac{1}{6}}}.\end{aligned}$$

(*ii*) The limit $\lim_{x\to 0} \frac{x^2}{1+x-e^x}$ is to be evaluated.
By differentiation we obtain

$$\frac{d}{dx}\left[x^2\right] = 2x \qquad \frac{d^2}{dx^2}\left[x^2\right] = 2$$

$$\frac{d}{dx}\left[1+x-e^x\right] = 1-e^x \qquad \frac{d^2}{dx^2}\left[1+x-e^x\right] = -e^x$$

Observe that:

(a) The first, and second order derivatives of x^2 and $1+x-e^x$ are all differentiable over $\mathbb{R}$.

(b) $\lim_{x\to 0} x^2 = \lim_{x\to 0} 2x = \lim_{x\to 0} 1 + x - e^x = \lim_{x\to 0} 1 - e^x = 0$.
(c) $2x \neq 0$ and $1 - e^x \neq 0$ for all $x \neq 0$.

Which shows that we meet the conditions for L'Hôpital's Rule (last condition to be proven below). Thus, applying L'Hôpital's Rule 2 times,

$$\begin{aligned}\lim_{x\to 0} \frac{x^2}{1+x-e^x} &= \lim_{x\to 0} \frac{2x}{1-e^x} \\ &= \lim_{x\to 0} \frac{2}{-e^x} \\ &= -\frac{2}{1} = \underline{\underline{-2}}.\end{aligned}$$

Exercise 5.

a) The critical points of the function defined by $f(x) = -x^2 + 2x + 2$ is to be found.
At critical points, $f'(x) = 0$. Therefore,

$$\begin{aligned}\frac{d}{dx}\left[-x^2+2x+2\right] &= 0 \\ -2x+2 &= 0 \\ x &= \underline{\underline{1}}\end{aligned}$$

Thus we conclude that critical points of $f = \underline{\underline{\{1\}}}$.

b) The critical points of the function defined by $f(x) = x^3 - 3$ is to be found.
At critical points, $f'(x) = 0$. Therefore,

$$\begin{aligned}\frac{d}{dx}\left[x^3-3\right] &= 0 \\ 3x^2 &= 0 \\ x &= \underline{\underline{0}}\end{aligned}$$

Thus we conclude that critical points of $f = \underline{\underline{\{0\}}}$.

c) The critical points of the function defined by $f(x) = \cos(x)$ is to be found.
At critical points, $f'(x) = 0$. Therefore,

$$\begin{aligned}\frac{d}{dx}\left[\cos(x)\right] &= 0 \\ -\sin(x) &= 0\end{aligned}$$

As such, for all x such that $\sin(x) = 0$, $\frac{d}{dx}\left[\cos(x)\right] = 0$.
Thus we conclude that critical points of $f = \underline{\underline{\{k\pi \mid k \in \mathbb{Z}\}}}$.

d) The critical points of the function defined by $f(x) = \sin(x) + \cos(x)$ is to be found.
At critical points, $f'(x) = 0$. Therefore,

$$\begin{aligned}\frac{d}{dx}\left[\sin(x)+\cos(x)\right] &= 0 \\ \cos(x) - \sin(x) &= 0 \\ \sin(x) &= \cos(x) \\ \tan(x) &= 1\end{aligned}$$

As such, for all x such that $\tan(x) = 1$, $\frac{d}{dx}\left[\sin(x)+\cos(x)\right] = 0$.
Thus we conclude that critical points of $f = \underline{\underline{\{\frac{\pi}{4} + k\pi \mid k \in \mathbb{Z}\}}}$.