
Solution to Homework 4 (4+2+4+2+3+3=18pts)

Exercise 1.

a.) The derivative of the function defined by $f(x) = 5x^4 + 4x^2 - 1$ is to be calculated.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[5x^4\right] + \frac{d}{dx}\left[4x^2\right] - \frac{d}{dx}[1] \\ &= \underline{\underline{20x^3 + 8x}}. \end{aligned}$$

b.) The derivative of the function defined by $f(x) = (x^5 + 1)(x^2 - 1)$ is to be calculated.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[x^5 + 1\right] \cdot (x^2 - 1) + (x^5 + 1) \cdot \frac{d}{dx}\left[x^2 - 1\right] \\ &= 5x^4 \cdot (x^2 - 1) + (x^5 + 1) \cdot 2x \\ &= 5x^6 - 5x^4 + 2x6 + 2x \\ &= \underline{\underline{7x^6 - 5x^4 + 2x}}. \end{aligned}$$

c.) The derivative of the function defined by $f(x) = \frac{5x-1}{x-5}$, $x \neq 5$ is to be calculated.

$$\begin{aligned} f'(x) &= \frac{\frac{d}{dx}[5x-1] \cdot (x-5) - \frac{d}{dx}[x-5] \cdot (5x-1)}{(x-5)^2} \\ &= \frac{5 \cdot (x-5) - 1 \cdot (5x-1)}{(x-5)^2} \\ &= \frac{5x - 25 - 5x + 1}{(x-5)^2} \\ &= \underline{\underline{-\frac{24}{(x-5)^2}}}. \end{aligned}$$

d.) The derivative of the function defined by $f(x) = \frac{x^{25}-2x}{x^2+3}$ is to be calculated.

$$\begin{aligned} f'(x) &= \frac{\frac{d}{dx}\left[x^{25} - 2x\right] \cdot (x^2+3) - \frac{d}{dx}\left[x^2+3\right] \cdot (x^{25} - 2x)}{(x^2+3)^2} \\ &= \frac{(25x^{24} - 2) \cdot (x^2+3) - 2x \cdot (x^{25} - 2x)}{(x^2+3)^2} \\ &= \frac{(25x^{26} - 2x^2 + 75x^{24} - 6) - (2x^{26} - 4x^2)}{(x^2+3)^2} \\ &= \underline{\underline{\frac{23x^{26} + 75x^{24} + 2x^2 - 6}{(x^2+3)^2}}}. \end{aligned}$$

Exercise 2.

Observe that $\forall x \in \mathbb{R}$, $\sin^2(x) + \cos^2(x) = 1$. Thus,

$$\begin{aligned} \frac{d}{dx}\left[\sin^2(x) + \cos^2(x)\right] &= \frac{d}{dx}[1] \\ \frac{d}{dx}\left[\sin^2(x)\right] + \frac{d}{dx}\left[\cos^2(x)\right] &= 0 \\ 2\sin(x)\frac{d}{dx}[\sin(x)] + 2\cos(x)\frac{d}{dx}[\cos(x)] &= 0 \end{aligned}$$

Here one uses $\frac{d}{dx}[\sin(x)] = \cos(x)$:

$$\begin{aligned} 2\sin(x)\cos(x) + 2\cos(x)\frac{d}{dx}[\cos(x)] &= 0 \\ 2\cos(x)\left(\sin(x) + \frac{d}{dx}[\cos(x)]\right) &= 0. \end{aligned}$$

Since $\cos(x)$ is a continuous function (but isn't the zero function), with a continuous derivative, one infers that

$$\sin(x) + \frac{d}{dx}[\cos(x)] = 0 \implies \frac{d}{dx}[\cos(x)] = \underline{\underline{-\sin(x)\,.}}$$

Exercise 3.

The function $f : \mathbb{R} \to \mathbb{R}$ is defined by:

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

1.) The continuity of f at $x = 0$ is to be proven.
Let $\varepsilon > 0$ be given. Choose $\delta < \sqrt{\varepsilon}$. Thus, for $|x| < \delta$,

$$\begin{aligned} |x - 0| < \delta &\Rightarrow |x| < \sqrt{\varepsilon} \\ &\Rightarrow |x^2| < \varepsilon \\ &\Rightarrow \left|\sin\left(\frac{1}{x}\right)\right| \cdot |x^2| < \varepsilon \qquad \left(\text{since } \left|\sin\left(\frac{1}{h}\right)\right| \leq 1\right) \\ &\Rightarrow \left|x^2 \sin\left(\frac{1}{x}\right) - 0\right| < \varepsilon \\ &\Rightarrow |f(x) - f(0)| < \varepsilon. \end{aligned}$$

This is equivalent to the equation

$$\lim_{x \to 0} f(x) = f(0).$$

From which one infers that $\underline{\underline{f \text{ is continuous at } x = 0.}}$

2.) By the definition of the derivative,

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} \\ &= \lim_{h \to 0} h \sin\left(\frac{1}{h}\right). \end{aligned}$$

One can then prove the existence of this limit.
For a given $\varepsilon > 0$, choose $\delta = \varepsilon$. For all $|h| < \delta$,

$$\begin{aligned} |h| < \delta &\Rightarrow |h| < \varepsilon \\ &\Rightarrow |h| \cdot \left|\sin\left(\frac{1}{h}\right)\right| < \varepsilon \qquad \text{since } \left|\sin\left(\frac{1}{x}\right)\right| \leq 1 \\ &\Rightarrow \left|h \sin\left(\frac{1}{h}\right) - 0\right| < \varepsilon \end{aligned}$$

Which shows that $\lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0$. Therefore,

$$f'(0) = \underline{\underline{0\,.}}$$

3.) For $x \neq 0$, one can find $f'(x)$ by:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[x^2 \sin\left(\frac{1}{x}\right)\right] \\ &= \frac{d}{dx}\left[x^2\right] \cdot \sin\left(\frac{1}{x}\right) + x^2 \cdot \frac{d}{dx}\left[\sin\left(\frac{1}{x}\right)\right]. \end{aligned}$$

Observe that $\sin\left(\frac{1}{x}\right)$ is a composition of functions. By the composition rule,

$$\begin{aligned} f'(x) &= 2x \cdot \sin\left(\frac{1}{x}\right) + x^2 \cdot \cos\left(\frac{1}{x}\right) \cdot \frac{d}{dx}\left[\frac{1}{x}\right] \\ &= 2x \sin\left(\frac{1}{x}\right) + x^2 \cos\left(\frac{1}{x}\right) \cdot \frac{-1}{x^2} \\ &= \underline{\underline{2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)}}. \end{aligned}$$

4.) (From the result of the previous section) we have that for all $x \in \mathbb{R}$

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

The expression at all $x \neq 0$ is well-defined (as sin and cos are continuous).
One then concludes that $\underline{\underline{f' \text{ is well-defined at all } x \in \mathbb{R}.}}$

Then, the continuity of f' at $x = 0$ is to be discussed.
Recall that the continuity of a function at a point x_0 implies

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Claim: $\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$ does not exist.

Proof by Contradiction. Suppose that this limit exists, and is equal to a number $L \in \mathbb{R}$.
If so, then for arbitrary $\varepsilon > 0$ $\exists \delta > 0$ such that

$$|x| < \delta \implies \left|\cos\left(\frac{1}{x}\right) - L\right| < \varepsilon$$

Set $\varepsilon = \frac{1}{2}$, and let its corresponding δ exist.
Define $N \in \mathbb{N}$ such that $\frac{1}{(2N+1)\pi} < \frac{1}{2N\pi} < \delta$. Then,

$$\begin{aligned} 2 &= |1 - (-1)| \\ &= |\cos(2N\pi) - \cos(\,(2N+1)\pi\,)| \\ &= \left|\cos\left(\frac{1}{\frac{1}{2N\pi}}\right) - \cos\left(\frac{1}{\frac{1}{(2N+1)\pi}}\right)\right| \\ &= \left|\cos\left(\frac{1}{\frac{1}{2N\pi}}\right) - L + L - \cos\left(\frac{1}{\frac{1}{(2N+1)\pi}}\right)\right|. \end{aligned}$$

By the Triangle Inequality $|x+y| \leq |x| + |y|$,

$$\begin{aligned} 2 &\leq \left|\cos\left(\frac{1}{\frac{1}{2N\pi}}\right) - L\right| + \left|L - \cos\left(\frac{1}{\frac{1}{(2N+1)\pi}}\right)\right| \\ &\leq \left|\cos\left(\frac{1}{\frac{1}{2N\pi}}\right) - L\right| + \left|\cos\left(\frac{1}{\frac{1}{(2N+1)\pi}}\right) - L\right| \\ &\leq \varepsilon + \varepsilon \\ 2 &\leq 2\varepsilon, \end{aligned}$$

which is a contradiction as $\varepsilon = \frac{1}{2}$. One then concludes that $\lim_{x\to 0} \cos\left(\frac{1}{x}\right)$ does not exist. As a direct consequence,

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

Also does not exist, and thus

$$\lim_{x\to 0} f'(x) \neq 0 = f'(0)$$

Which means that the derivative of f is not continuous at $x = 0$.

Exercise 4.

1. The value of $\lim_{h\to 0} \frac{\cos(h)-1}{h}$ is to be evaluated.

$$\begin{aligned} \lim_{h\to 0} \frac{\cos(h)-1}{h} &= \lim_{h\to 0} \left(\frac{\cos(h)-1}{h} \cdot \frac{\cos(h)+1}{\cos(h)+1}\right) \\ &= \lim_{h\to 0} \left(\frac{\cos^2(h)-1}{h\cdot(\cos(h)+1)}\right) \\ &= \lim_{h\to 0} \left(\frac{-\sin^2(h)}{h\cdot(\cos(h)+1)}\right) \\ &= -\lim_{h\to 0} \left(\frac{\sin(h)}{h}\right) \cdot \lim_{h\to 0} \left(\frac{\sin(h)}{\cdot(\cos(h)+1)}\right) \\ &= -1 \cdot \frac{0}{2} = 0\,. \end{aligned}$$

2. The value of $\lim_{h\to 0} \frac{\cos(h)-1}{h^2}$ is to be evaluated.

$$\begin{aligned} \lim_{h\to 0} \frac{\cos(h)-1}{h^2} &= \lim_{h\to 0} \left(\frac{\cos(h)-1}{h^2} \cdot \frac{\cos(h)+1}{\cos(h)+1}\right) \\ &= \lim_{h\to 0} \left(\frac{\cos^2(h)-1}{h^2\cdot(\cos(h)+1)}\right) \\ &= \lim_{h\to 0} \left(\frac{-\sin^2(h)}{h^2\cdot(\cos(h)+1)}\right) \\ &= -\lim_{h\to 0} \left(\frac{\sin(h)}{h}\right) \cdot \lim_{h\to 0} \left(\frac{\sin(h)}{h}\right) \cdot \lim_{h\to 0} \left(\frac{1}{(\cos(h)+1)}\right) \\ &= -1 \cdot 1 \cdot \frac{1}{2} = -\frac{1}{2}\,. \end{aligned}$$

Exercise 5.

For $x \in (-1,0) \cup (0,1)$ let $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ and $g(x) = \sin(x)$

a.) First, we find the expression for $f'(x)$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[x^2\right] \cdot \sin\left(\frac{1}{x}\right) + x^2 \cdot \frac{d}{dx}\left[\sin\left(\frac{1}{x}\right)\right] \\ &= 2x \cdot \sin\left(\frac{1}{x}\right) + x^2 \cdot \cos\left(\frac{1}{x}\right) \cdot \frac{-1}{x^2} \\ &= 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \end{aligned}$$

And by basic differentiation we obtain $g'(x) = \cos(x)$.

The existence of the limit $\lim_{x\to 0} \frac{f(x)}{g(x)}$ is to be proven.

$$\begin{aligned} \lim_{x\to 0} \frac{f(x)}{g(x)} &= \lim_{x\to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{\sin(x)} \\ &= \lim_{x\to 0} \frac{x}{\sin(x)} \cdot \lim_{x\to 0} x \sin\left(\frac{1}{x}\right) \end{aligned}$$

$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, and by the result of <u>Exercise 3</u> part 2.), $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$,

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{1}{1} \cdot 0 = \underline{\underline{0\,.}}$$

On the other hand, the limit $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ can be expressed as

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{2x \sin\left(\frac{1}{x}\right)}{\cos(x)} - \lim_{x\to 0} \frac{\cos\left(\frac{1}{x}\right)}{\cos(x)}$$

The first term is well-defined (by the result of <u>Exercise 3</u> part 2 and since $\cos(0) = 1$), while the second limit does not exist (by the result of <u>Exercise 3</u> part 4).
Therefore, one concludes that $\underline{\underline{\lim_{x\to 0} \frac{f'(x)}{g'(x)} \text{ does not exist.}}}$

b.) This situation does not contradict L'Hospital's rule in any way since L'Hospital's rule says

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$$

if the latter limit exists. However, it does not say anything about the first limit if the latter limit does not exist (which is the case in part a.)).

Exercise 6.

a) The 1^{st} derivative of the function defined by $f(x) = \cos(x)$ is to be calculated.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\cos(x)\right] \\ &= \underline{\underline{-\sin(x)}}\,. \end{aligned}$$

The 2^{nd} derivative of the function defined by $f(x) = \cos(x)$ is to be calculated.

$$\begin{aligned} f^{(2)}(x) &= \frac{d}{dx}[-\sin(x)] \\ &= -\frac{d}{dx}[\sin(x)] \\ &= \underline{\underline{-\cos(x)}}\,. \end{aligned}$$

The 3^{rd} derivative of the function defined by $f(x) = \cos(x)$ is to be calculated.

$$\begin{aligned} f^{(3)}(x) &= \frac{d}{dx}[-\cos(x)] \\ &= -\frac{d}{dx}[\cos(x)] \\ &= -(-\sin(x)) \\ &= \underline{\underline{\sin(x)}}\,. \end{aligned}$$

b) The 1^{st} derivative of the function defined by $f(x) = \sin(x)\cos(x)$ is to be calculated.

$$\begin{aligned} f'(x) &= \frac{d}{dx}[\sin(x)\cos(x)] \\ &= \frac{d}{dx}[\sin(x)] \cdot \cos(x) + \sin(x) \cdot \frac{d}{dx}[\cos(x)] \\ &= \cos(x) \cdot \cos(x) + \sin(x) \cdot (-\sin(x)) \\ &= \underline{\underline{\cos^2(x) - \sin^2(x)}}. \end{aligned}$$

The 2^{nd} derivative of the function defined by $f(x) = \sin(x)\cos(x)$ is to be calculated.

$$\begin{aligned} f^{(2)}(x) &= \frac{d}{dx}\left[\cos^2(x) - \sin^2(x)\right] \\ &= 2\cos(x)\frac{d}{dx}[\cos(x)] - 2\sin(x)\frac{d}{dx}[\sin(x)] \\ &= 2\cos(x) \cdot (-\sin(x)) - 2\sin(x) \cdot \cos(x) \\ &= \underline{\underline{-4\sin(x)\cos(x)}}. \end{aligned}$$

The 3^{rd} derivative of the function defined by $f(x) = \sin(x)\cos(x)$ is to be calculated.

$$\begin{aligned} f^{(3)}(x) &= \frac{d}{dx}[-4\sin(x)\cos(x)] \\ &= -4 \cdot \frac{d}{dx}[\sin(x)\cos(x)] \\ &= (-4) \cdot (\cos^2(x) - \sin^2(x)) \\ &= \underline{\underline{4\sin^2(x) - 4\cos^2(x)}}. \end{aligned}$$

c) The 1^{st} derivative of the function defined by $f(x) = x^4 + x^3 + x^2 + x + 1$ is to be calculated.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[x^4 + x^3 + x^2 + x + 1\right] \\ &= 4 \cdot x^3 + 3 \cdot x^2 + 2 \cdot x + 1 \cdot 1 + 0 \\ &= \underline{\underline{4x^3 + 3x^2 + 2x + 1}}. \end{aligned}$$

The 2^{nd} derivative of the function defined by $f(x) = x^4 + x^3 + x^2 + x + 1$ is to be calculated.

$$\begin{aligned} f^{(2)}(x) &= \frac{d}{dx}\left[4x^3 + 3x^2 + 2x + 1\right] \\ &= 4 \cdot 3 \cdot x^2 + 3 \cdot 2 \cdot x + 2 \cdot 1 + 0 \\ &= \underline{\underline{12x^2 + 6x + 2}}. \end{aligned}$$

The 3^{rd} derivative of the function defined by $f(x) = x^4 + x^3 + x^2 + x + 1$ is to be calculated.

$$\begin{aligned} f^{(3)}(x) &= \frac{d}{dx}\left[12x^2 + 6x + 2\right] \\ &= 12 \cdot 2 \cdot x + 6 \cdot 1 + 0 \\ &= \underline{\underline{24x + 6}}. \end{aligned}$$