Solution to Homework 3 (2+2+4+4+3=15pts)

Exercise 1.

For $I \subset \mathbb{R}$ we define a continuous function $f : I \to \mathbb{R}$ defined by $f(x)$. For a certain $x \in I$ set $f(x) \neq 0$.

Claim: $\exists\, \delta > 0$ such that $\forall h,\ |h| \leq \delta,\ f(x+h) \neq 0$.

Continuity of f implies that

$$\forall\ \varepsilon > 0\ \exists\, \delta > 0 \text{ such that } |h| < \delta \implies f(x+h) - f(x) < \varepsilon.$$

There are two possible cases, depending whether $f(x) > 0$ or $f(x) < 0$.

If $f(x) > 0$, set $\varepsilon > 0$ such that $0 < \varepsilon < f(x)$. For $|h| < \delta$,

$$\begin{aligned} & |f(x+h) - f(x)| \ < \ \varepsilon \\ \Rightarrow\ & |f(x+h) - f(x)| \ < \ f(x) \\ \Leftrightarrow\ & -f(x) \ < \ f(x+h) - f(x) \ < \ f(x) \\ \Leftrightarrow\ & 0 \ < \ f(x+h) \ < \ 2f(x) \end{aligned}$$

If $f(x) < 0$, set $\varepsilon > 0$ such that $0 < \varepsilon < -f(x)$. For $|h| < \delta$,

$$\begin{aligned} & |f(x+h) - f(x)| \ < \ \varepsilon \\ \Rightarrow\ & |f(x+h) - f(x)| \ < \ -f(x) \\ \Leftrightarrow\ & f(x) \ < \ f(x+h) - f(x) \ < \ -f(x) \\ \Leftrightarrow\ & 2f(x) \ < \ f(x+h) \ < \ 0 \end{aligned}$$

We conclude then that $\exists\, \delta > 0$ such that $f(x+h) \neq 0\ \ \forall h \in [-\delta, \delta]$. **Q.E.D.**

Exercise 2.

The slope of $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x^2 - 3x + 2$ at x can be found by the definition of the derivative:

$$\begin{aligned} m &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(2(x+h)^2 - 3(x+h) + 2) - (2x^2 - 3x + 2)}{h} \\ &= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 3x - 3h + 2 - 2x^2 + 3x - 2}{h} \\ &= \lim_{h \to 0} \frac{4xh + 2h^2 - 3h}{h} \\ &= \lim_{h \to 0} (\, 4x + 2h - 3\,) \\ &= 4x - 3. \end{aligned}$$

Let x_0 be a fixed arbitrary point on $\mathbb{R}$. We find the equation of the tangent line:

$$\begin{aligned} y &= m(x - x_0) + f(x_0) \\ &= (4x_0 - 3)(x - x_0) + 2x_0^2 - 3x_0 + 2 \\ &= 4xx_0 - 3x - 4x_0^2 + 3x_0 + 2x_0^2 - 3x_0 + 2 \\ &= (4x_0 - 3)x - 2x_0^2 + 2. \end{aligned}$$

Exercise 3.

1.) Fix $x \in \mathbb{R}$. Choose $N \in \mathbb{N}$ so that $N > |x|$. This implies that $\left|\frac{x}{N}\right| < 1$. Observe that:

$$f(x) = \sum_{n=0}^{N} \frac{x^n}{n!} + \sum_{n=N+1}^{\infty} \frac{x^n}{n!}$$

The first term of this expression is a finite sum, and thus evaluates to a finite number.

Consider the second term:

$$\begin{aligned}
\left|\sum_{n=N+1}^{\infty} \frac{x^n}{n!}\right| &= \left|\sum_{n=N+1}^{\infty} \frac{x^n}{(n)\cdot(n-1)\cdot(n-2)...(N+1)\cdot(N)\cdot(N-1)!}\right| \\
&\leq \sum_{n=N+1}^{\infty} \frac{|x|^n}{(n)\cdot(n-1)\cdot(n-2)...(N+1)\cdot(N)\cdot(N-1)!} \\
&\leq \sum_{n=N+1}^{\infty} \frac{|x|}{n}\cdot\frac{|x|}{n-1}\ \cdots\ \frac{|x|}{N}\cdot\frac{|x|^{N-1}}{(N-1)!} \\
&\leq \sum_{n=N+1}^{\infty} \frac{|x|^{n-N+1}}{N^{n-N+1}}\cdot\frac{|x|^{N-1}}{N^{N-1}}\cdot\frac{N^{N-1}}{(N-1)!} \\
&\leq \frac{N^{N-1}}{(N-1)!}\sum_{n=N+1}^{\infty}\left|\frac{x}{N}\right|^n \leq \frac{N^{N-1}}{(N-1)!}\sum_{n=0}^{\infty}\left|\frac{x}{N}\right|^n
\end{aligned}$$

Recall that $\left|\frac{x}{N}\right| < 1$. So by the sum of a geometric series,

$$\sum_{n=N+1}^{\infty} \frac{x^n}{n!} \leq \frac{N^{N-1}}{(N-1)!}\cdot\frac{1}{1-\left|\frac{x}{N}\right|} = \frac{N^{N-1}}{(N-1)!}\cdot\frac{N}{N-|x|} < \infty$$

Therefore, we conclude

$$f(x) = \sum_{n=0}^{N} \frac{x^n}{n!} + \sum_{n=N+1}^{\infty} \frac{x^n}{n!} < \infty$$

Which shows that the sum is convergent.

2.) Using the definition of the derivative,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sum_{n=0}^{\infty} \frac{(x+h)^n}{n\,!} - \sum_{n=0}^{\infty} \frac{(x)^n}{n\,!}}{h} \\
&= \lim_{h\to 0} \frac{\sum_{n=0}^{\infty} \frac{(x+h)^n - x^n}{n\,!}}{h} \\
&= \lim_{h\to 0} \frac{\frac{(x+h)^0 - x^0}{0\,!} + \sum_{n=1}^{\infty} \frac{(x+h)^n - x^n}{n\,!}}{h} \\
&= \lim_{h\to 0} \frac{0 + \sum_{n=1}^{\infty} \frac{(x+h)^n - x^n}{n\,!}}{h} \\
&= \sum_{n=1}^{\infty} \frac{\lim_{h\to 0} \frac{(x+h)^n - x^n}{h}}{n!} \\
&= \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n\,!} \\
&= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)\,!} \\
&= \sum_{u=0}^{\infty} \frac{x^u}{u\,!} = \underline{\underline{f(x)}} \qquad \text{(Change of Variable to } u := n-1\text{)}
\end{aligned}$$

Remark: Generally, one can not interchange the position of a limit and an infinite sum (the step with the red equal sign). In this case, it is justifiable, but the proof is not included in this document.

3.) The slope of the tangent line to a point in the graph of $f(x)$ is equal to the value of $f(x)$ at that point in the graph. Also, it follows that the value of $f(x)$ goes to $+\infty$ as $x \to \infty$.

Remark: This interesting function is known as the (natural) exponential function.

Exercise 4.

Let the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous functions.
Claim: for all $\lambda \in \mathbb{R}$, the function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = \lambda f(x) + g(x)$$

is continuous over $\mathbb{R}$.

If $\lambda = 0$, the function $h = \lambda f + g$ simplifies to the function g, which is continuous.
Therefore, consider the cases where $\lambda \neq 0$.
The continuity of f at a fixed x implies that for arbitrary $\varepsilon > 0$ there exists δ_1 such that

$$|h| < \delta_1 \Longrightarrow |f(x+h) - f(x)| < \frac{\varepsilon}{2|\lambda|}$$

The continuity of g at x implies that for arbitrary $\varepsilon > 0$ there exists δ_2 such that

$$|h| < \delta_2 \Longrightarrow |g(x+h) - g(x)| < \frac{\varepsilon}{2}$$

For a given $\varepsilon > 0$, set $\delta = \min\{\delta_1, \delta_2\}$. Thus, for $|h| < \delta$,

$$\begin{aligned}
|h(x+h) - h(x)| &= |\lambda f(x+h) + g(x+h) - \lambda f(x) - g(x)| \\
&\leq |\lambda f(x+h) - \lambda f(x)| + |g(x+h) - g(x)| \\
&\leq |\lambda||f(x+h) - f(x)| + |g(x+h) - g(x)| \\
&\leq |\lambda| \frac{\varepsilon}{2|\lambda|} + \frac{\varepsilon}{2} \\
&= \varepsilon,
\end{aligned}$$

which proves that h is continuous at x. Since x is arbitrary, this proves the claim.
Claim: the function $j : \mathbb{R} \to \mathbb{R}$ defined by

$$j(x) = f(x)g(x)$$

is continuous over $\mathbb{R}$.

Let $x \in \mathbb{R}$ be fixed and set

$$M := \max\Big\{ \sup_{t \in [x-1, x+1]} |f(t)|, \sup_{t \in [x-1, x+1]} |g(t)| \Big\}$$

If $M = 0$, we conclude that fg is continuous as it is the constant zero function near x.
Then, we consider the case when $M \neq 0$:
The continuity of f at x implies that for arbitrary $\varepsilon > 0$ there exists δ_1 such that

$$|h| < \delta_1 \Longrightarrow |f(x+h) - f(x)| < \frac{\varepsilon}{2M}$$

The continuity of g at x implies that for arbitrary $\varepsilon > 0$ there exists δ_2 such that

$$|h| < \delta_2 \Longrightarrow |g(x+h) - g(x)| < \frac{\varepsilon}{2M}$$

For a given $\varepsilon > 0$, set $\delta = \min\{\delta_1, \delta_2, 1\}$. Thus, for $|h| < \delta$,

$$\begin{aligned}
|j(x+h) - j(x)| &= |f(x+h)g(x+h) - f(x)g(x)| \\
&= |f(x+h)g(x+h) + f(x+h)g(x) - f(x+h)g(x) - f(x)g(x)| \\
&\leq |f(x+h)g(x+h) - f(x+h)g(x)| + |f(x+h)g(x) - f(x)g(x)| \\
&\leq |f(x+h)| \cdot |g(x+h) - g(x)| + |g(x)| \cdot |f(x+h) - f(x)| \\
&\leq M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} \\
&= \varepsilon,
\end{aligned}$$

which proves that j is continuous at x. Since x is arbitrary, this proves the claim.

Exercise 5.

1.) The uniform continuity of the function $f_1 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = x$ is to be discussed.
Let $\varepsilon > 0$ and $x \in \mathbb{R}$ be given. Choose $\delta = \varepsilon$. Therefore, $\forall h,\ |h| < \delta$, one has:

$$\begin{aligned}
|h| < \delta &\Rightarrow |h| < \varepsilon \\
&\Rightarrow |x + h - x| < \varepsilon \\
&\Rightarrow |f_1(x+h) - f_1(x)| < \varepsilon
\end{aligned}$$

Which proves that the continuity of f_1. As δ is independent of x, f_1 is uniformly continuous.

2.) The uniform continuity of the function $f_2 : \mathbb{R} \to \mathbb{R}$ defined by $f_2(x) = x^2$ is to be discussed.

Proof By Contradiction. Suppose that the function f_2 is uniformly continuous.

If so, for arbitrary $\varepsilon > 0$, there must exist $\delta > 0$ such that $\forall x \in \mathbb{R}$

$$|h| < \delta \implies |(x+h)^2 - x^2| < \varepsilon$$

Let $\varepsilon > 0$ and its corresponding $\delta > 0$ be given. Choose $h = \frac{\delta}{2}$.

If the function is uniformly continuous, the inequality

$$\left| \left(x + \frac{\delta}{2} \right)^2 - x^2 \right| = \left| \delta x + \frac{\delta^2}{4} \right| < \varepsilon$$

holds for all $x \in \mathbb{R}$. However, if we consider $x = \frac{\varepsilon}{\delta}$, the inequality simplifies to

$$\left| \varepsilon + \frac{\delta^2}{4} \right| < \varepsilon$$

Which is clearly false, as $\varepsilon, \delta > 0$.

Therefore, we conclude that f_2 is not uniformly continuous.