Nagoya University, G30 program — Fall 2024

Mathematics Tutorial Ia — Instructor : Serge Richard

Homework 2

Exercise 1 *A function $f : \mathbb{R} \to \mathbb{R}$ is said to be* even *if $f(-x) = f(x)$ for any $x \in \mathbb{R}$, and f is said to be* odd *if $f(-x) = -f(x)$ for any $x \in \mathbb{R}$.*
1) Determine which of the functions defined for $x \in \mathbb{R}$ by

$$a)\ f(x) = x, \quad b)\ f(x) = x^2, \quad c)\ f(x) = x^2 + x, \quad d)\ f(x) = \sin(x), \quad e)\ f(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

are even or odd ?
2) Show that for any function f, the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \frac{1}{2}\big(f(x) + f(-x)\big)$ is an even function while the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = \frac{1}{2}\big(f(x) - f(-x)\big)$ is an odd function. In addition, observe that $f = g + h$.
3) What can you say about the graph of an even function, and about the graph of an odd function ?

Exercise 2 *Determine the equation of the function $f : \mathbb{R} \to \mathbb{R}$ whose graph is a straight line containing the points (x_1, y_1) and (x_2, y_2) of $\mathbb{R}^2$. What is the slope of this line ?*

Exercise 3 *Compute the following limits, if they exist:*

1. $\lim_{x\to 0_-} \left(\frac{1}{x} + \frac{1}{|x|}\right)$ *and* $\lim_{x\to 0_+} \left(\frac{1}{x} + \frac{1}{|x|}\right)$,
2. $\lim_{x\to 2_+} \frac{x^2+x-6}{|x-2|}$ *and* $\lim_{x\to 2_-} \frac{x^2+x-6}{|x-2|}$,
3. $\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h}$.

Exercise 4 *Show as precisely as possible that the following limits exist:*

$$\lim_{x\to 0} x^n = 0 \qquad \text{for any } n \in \mathbb{N},$$
$$\lim_{x\to 0} x \sin(1/x) = 0.$$

On the other hand, show that the function

$$(0, \infty) \ni x \mapsto \sin(1/x) \in \mathbb{R}$$

has no limit at 0 from the right.

Exercise 5 *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ converging to a_∞, and let $(b_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ converging to b_∞. Let also $\lambda \in \mathbb{R}$. Show that the following properties hold:*

(i) *The sequence $(\lambda a_n)_{n\in\mathbb{N}}$ converges to λa_∞,*

(ii) *The sequence $(a_n + b_n)_{n\in\mathbb{N}}$ converges to $a_\infty + b_\infty$,*

(iii) *The sequence $(a_n b_n)_{n\in\mathbb{N}}$ converges to $a_\infty b_\infty$.*