Solution to Homework 1: (2+3+6=11pts)

Exercise 1.

Consider a sequence $(a_n)_{n\in\mathbb{N}}$ that is convergent to 0 as $n\to\infty$ and a sequence $(b_n)_{n\in\mathbb{N}}$ that is bounded by a fixed $C>0$.

The convergence of $(a_n)_{n\in\mathbb{N}}$ to 0 implies that

$$\text{for } \varepsilon>0\,,\ \exists N\in\mathbb{N} \text{ such that for } n\geq N\,,\ |a_n|<\varepsilon$$

Therefore, $\exists N$ such that $\forall n\geq N$, $|a_n|<\frac{\varepsilon}{C}$.

$$\begin{aligned}|a_n\cdot b_n| &= |a_n|\cdot|b_n|\\ &\leq |a_n|\cdot C\\ &\leq \frac{\varepsilon}{C}\cdot C\\ &\leq \varepsilon,\end{aligned}$$

which proves that $(a_nb_n)_{n\in\mathbb{N}}$ converges to 0. As such, $\lim_{n\to\infty} a_nb_n=0$.

Exercise 2.

In this exercise, we rely on the Monotone Convergence Theorem (**Theorem 1.6.**). If an increasing sequence is upper-bounded, then it is convergent.

This statement can be proven by induction.

Base Case: For n = 1,

$$a_2=2>1=a_1$$

Which proves the base case.

Inductive Hypothesis: For a fixed $n\in\mathbb{N}$, and for all $m\in\mathbb{N}$, $m<n$, $a_{m+1}-a_m>0$.

Inductive Step: Consider the expression $a_{n+1}-a_n$:

$$\begin{aligned}a_{n+1}-a_n &= 3-\frac{1}{a_n}-\left(3-\frac{1}{a_{n-1}}\right)\\ &= \frac{1}{a_{n-1}}-\frac{1}{a_n}\end{aligned}$$

By the Inductive Hypothesis, we get that $a_{n-1}<a_n$, so $\frac{1}{a_{n-1}}>\frac{1}{a_n}$:

$$a_{n+1}-a_n = \frac{1}{a_{n-1}}-\frac{1}{a_n}>0,$$

which proves that the sequence is increasing, and thus, all $a_n>0$.

One then observes that since

$$a_n=3-\frac{1}{a_{n-1}},$$

we conclude that $a_n<3$ ($(a_n)_{n\in\mathbb{N}}$ is bounded).

Therefore, by the monotone convergence theorem, $(a_n)_{n\in\mathbb{N}}$ is convergent.

Let a_∞ denote the value that the sequence converges to as $n \to \infty$. Thus, a_∞ satisfies

$$\begin{aligned} a_\infty &= 3 - \frac{1}{a_\infty} \\ \Rightarrow a_\infty^2 &= 3a_\infty - 1 \\ \Rightarrow 0 &= a_\infty^2 - 3a_\infty + 1. \end{aligned}$$

We can then use the quadratic formula to find:

$$\begin{aligned} a_\infty &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(1)}}{2(1)} \\ &= \frac{3 \pm \sqrt{5}}{2}. \end{aligned}$$

We observe that $\frac{3-\sqrt{5}}{2} < 1$, so $a_\infty \neq \frac{3-\sqrt{5}}{2}$. Thus,

$$a_\infty = \underline{\underline{\frac{3+\sqrt{5}}{2} \approx 2.618}}$$

Exercise 3.

i) **Claim**: The sequence $(a_n)_{n\in\mathbb{N}^*}$ defined by $a_n = \frac{1}{n^2}$ is convergent.

For an arbitrary $\varepsilon > 0$, choose $N > \frac{1}{\sqrt{\varepsilon}}$, and let $n \geq N$, for $n, N \in \mathbb{N}$.

$$\begin{aligned} &N > \frac{1}{\sqrt{\varepsilon}} \\ \Rightarrow\ &\frac{1}{N} < \sqrt{\varepsilon} \\ \Rightarrow\ &\frac{1}{N^2} < \varepsilon \\ \Rightarrow\ &\frac{1}{n^2} < \varepsilon. \end{aligned}$$

Then, since $n \in \mathbb{N} \Longrightarrow n^2 > 0 \Longleftrightarrow \frac{1}{n^2} > 0$, one infers that $\left|\frac{1}{n^2}\right| = \frac{1}{n^2}$, and then

$$\begin{aligned} \left|\frac{1}{n^2} - 0\right| = \left|\frac{1}{n^2}\right| &< \varepsilon \\ \Rightarrow\ |a_n - 0| &< \varepsilon. \end{aligned}$$

This is in the form that shows the sequence $(a_n)_{n\in\mathbb{N}^*}$ converges to a value a_∞, here with $a_\infty = 0$, as $n \to \infty$.

In this case, the sequence $(a_n)_{n\in\mathbb{N}^*}$ defined by $a_n = \frac{1}{n^2}$ converges to $a_\infty = 0$.

ii) **Claim:** The sequence $(a_n)_{n\in\mathbb{N}^*}$ defined by $a_n = \sqrt{n+1} - \sqrt{n}$ is convergent

For an arbitrary $\varepsilon > 0$, choose $N > \frac{1}{4\varepsilon^2}$, and let $n \geq N$, for $n, N \in \mathbb{N}$.

$$\begin{aligned}
& N > \frac{1}{4\varepsilon^2} \\
\Rightarrow\ & \frac{1}{4N} < \varepsilon^2 \\
\Rightarrow\ & \frac{1}{2\sqrt{N}} < \sqrt{\varepsilon^2} \\
\Rightarrow\ & \frac{1}{2\sqrt{n}} \leq \frac{1}{2\sqrt{N}} < \varepsilon \\
\Rightarrow\ & \frac{1}{\sqrt{n}+\sqrt{n}} < \varepsilon \\
\Rightarrow\ & \frac{1}{\sqrt{n}+\sqrt{n+1}} < \varepsilon \\
\Rightarrow\ & \frac{1}{\sqrt{n}+\sqrt{n+1}} \cdot \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n}} < \varepsilon \\
\Rightarrow\ & \frac{\sqrt{n+1}-\sqrt{n}}{n+1-n} < \varepsilon \\
\Rightarrow\ & \frac{\sqrt{n+1}-\sqrt{n}}{1} < \varepsilon \\
\Rightarrow\ & \sqrt{n+1}-\sqrt{n} < \varepsilon.
\end{aligned}$$

Then, since $n \in \mathbb{N} \Longrightarrow \sqrt{n+1} > \sqrt{n} \Longleftrightarrow \sqrt{n+1}-\sqrt{n} > 0$, one has $\sqrt{n+1}-\sqrt{n} = |\sqrt{n+1}-\sqrt{n}|$ and

$$\begin{aligned}
& |\sqrt{n+1}-\sqrt{n}| < \varepsilon \\
\Rightarrow\ & |\sqrt{n+1}-\sqrt{n}-0| < \varepsilon \\
\Rightarrow\ & |a_n - 0| < \varepsilon.
\end{aligned}$$

This is in the form that shows that $(a_n)_{n\in\mathbb{N}}$ converges to a value a_∞.

In this case, the sequence $(a_n)_{n\in\mathbb{N}}$ defined by $a_n = \sqrt{n+1}-\sqrt{n}$ converges to $a_\infty = 0$.

iii) **Claim:** $(a_n)_{n\in\mathbb{N}}$ defined by $a_n = \sqrt{n^2+5n} - n$ is convergent.

For an arbitrary $\varepsilon > 0$, choose $N > \frac{25}{4\varepsilon}$, and let $n \geq N$, for $n, N \in \mathbb{N}$.

$$\begin{aligned}
& N > \frac{25}{4\varepsilon} \\
\Rightarrow\ & \frac{1}{N} < \frac{4\varepsilon}{25} \\
\Rightarrow\ & \frac{25}{4N} < \varepsilon \\
\Rightarrow\ & \frac{25}{4n} = \frac{5}{2} \cdot \frac{5}{2n} = \frac{5}{2} \cdot \left|\frac{5}{2n}\right| < \varepsilon.
\end{aligned}$$

One notices that since $n \in \mathbb{N}$, $1 + \sqrt{1 + \frac{5}{n}} > 2$. Thus,

$$\Rightarrow \frac{5}{2} \cdot \left| \frac{5/n}{1 + \sqrt{1 + 5/n}} \right| < \varepsilon$$

$$\Rightarrow \frac{5}{2} \cdot \left| \frac{-5/n}{1 + \sqrt{1 + 5/n}} \right| < \varepsilon$$

$$\Rightarrow \frac{5}{2} \cdot \left| \frac{-5/n}{\left(1 + \sqrt{1 + 5/n}\right)^2} \right| < \varepsilon$$

$$\Rightarrow \frac{5}{2} \cdot \left| \frac{1 - (1 + 5/n)}{\left(1 + \sqrt{1 + 5/n}\right)^2} \right| < \varepsilon$$

$$\Rightarrow \frac{5}{2} \cdot \left| \frac{(1 + \sqrt{1 + 5/n})(1 - \sqrt{1 + 5/n})}{\left(1 + \sqrt{1 + 5/n}\right)^2} \right| < \varepsilon$$

$$\Rightarrow \frac{5}{2} \cdot \left| \frac{1 - \sqrt{1 + 5/n}}{1 + \sqrt{1 + 5/n}} \right| < \varepsilon$$

$$\Rightarrow 5 \cdot \left| \frac{2 - \left(1 + \sqrt{1 + 5/n}\right)}{2\left(1 + \sqrt{1 + 5/n}\right)} \right| < \varepsilon$$

$$\Rightarrow 5 \cdot \left| \frac{1}{1 + \sqrt{1 + 5/n}} - \frac{1}{2} \right| < \varepsilon$$

$$\Rightarrow \left| \frac{5}{1 + \sqrt{1 + 5/n}} - \frac{5}{2} \right| < \varepsilon$$

$$\Rightarrow \left| \frac{5n}{n + \sqrt{n^2 + 5n}} - \frac{5}{2} \right| < \varepsilon$$

$$\Rightarrow \left| \frac{(n^2 + 5n) - n^2}{n + \sqrt{n^2 + 5n}} - \frac{5}{2} \right| < \varepsilon$$

$$\Rightarrow \left| \frac{(\sqrt{n^2 + 5n} - n)\cancel{(\sqrt{n^2 + 5n} + n)}}{\cancel{n + \sqrt{n^2 + 5n}}} - \frac{5}{2} \right| < \varepsilon$$

$$\Rightarrow \left| \left(\sqrt{n^2 + 5n} - n\right) - \frac{5}{2} \right| < \varepsilon$$

$$\Rightarrow \left| a_n - \frac{5}{2} \right| < \varepsilon$$

This is in the form that shows that $(a_n)_{n\in\mathbb{N}}$ converges to a value a_∞.
In this case, the sequence $(a_n)_{n\in\mathbb{N}}$ defined by $a_n = \sqrt{n^2 + 5n} - n$ converges to $a_\infty = \frac{5}{2}$.

Challenge Question.

Consider the sequence $(a_n)_{n\in\mathbb{N}^*}$ defined by $a_n = \left(1 + \frac{1}{n}\right)^n$.

By the binomial expansion, which can be defined as:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

a_n can be rewritten as:

$$a_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} (1)^{n-k}(n^{-1})^k = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}(1)(n^{-k}) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\frac{1}{n^k}$$

Expanding the terms gives:

$$\begin{aligned} a_n &= 1 + \frac{n}{n} + \frac{n(n-1)}{n^2}\cdot\frac{1}{2!} + \frac{n(n-1)(n-2)}{n^3}\cdot\frac{1}{3!} + \cdots + \frac{1}{n!}\frac{n(n-1)(n-2)...(1)}{n^n} \\ &= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)...\left(1-\frac{n-1}{n}\right). \end{aligned}$$

As $n \in \mathbb{N}$, $0 < 1 - \frac{j}{n} < 1 - \frac{j}{n+1}$ for all $j \in \mathbb{N}$ such that $j < n$, one gets

$$\begin{aligned} a_n &< 2 + \frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + ... + \frac{1}{n!}\left(1-\frac{1}{n+1}\right)...\left(1-\frac{n-1}{n+1}\right) \\ &< 2 + \frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \cdots + \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)...\left(1-\frac{n-1}{n+1}\right)\left(1-\frac{n}{n+1}\right). \end{aligned}$$

The right hand side is, in fact, the $(n+1)^{th}$ term of the sequence, a_{n+1}. Therefore,

$$a_n < a_{n+1}$$

Which proves that the sequence is increasing.

Next, one observes that

$$a_n < 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

Observe that $n! \geq 2^{n-1}$. Thus,

$$\begin{aligned} a_n &< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} \\ &< 1 + 2 - \frac{1}{2^{n-1}} \\ &< 3 - \frac{1}{2^{n-1}} \end{aligned}$$

As $\frac{1}{2^{n-1}} > 0$, we have that $a_n < 3$, which implies a_n is bounded.

Because $(a_n)_{n\in\mathbb{N}^*}$ is an increasing sequence that is bounded from above, by **Theorem 1.6.** (Monotone Convergence Theorem), we conclude that $(a_n)_{n\in\mathbb{N}^*}$ is convergent.