SML - Report

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Excersize 1.2.5

For $\sigma > 0$ and $\bar{x} \in \mathbb{R}$, set $\pi : \mathbb{R} \to [0, \infty)$ by

$$\pi(x) := \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\bar{x})^2\right).$$

Check that $\int \pi(x)\,dx = 1$.

$$\int \pi(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma} \int \exp\left(-\frac{1}{2\sigma^2}(x^2 - 2\bar{x}x + \bar{x}^2)\right) dx$$

$$\int \exp\left(-\frac{1}{2\sigma^2}(x^2 - 2\bar{x}x + \bar{x}^2)\right) dx = \exp\left(-\frac{\bar{x}^2}{2\sigma^2}\right) \int \exp\left(-\frac{x^2 - 2\bar{x}x}{2\sigma^2}\right) dx,$$

$$\int \exp\left(-\frac{x^2 - 2\bar{x}x}{2\sigma^2}\right) = \int \exp\left(-\frac{x^2}{2\sigma^2} + \frac{\bar{x}}{\sigma^2}x\right) dx$$

$$= \int \exp\left(\frac{\bar{x}^2}{2\sigma^2} - \left(\frac{x}{\sqrt{2}\sigma} - \frac{\bar{x}}{\sqrt{2}\sigma}\right)^2\right) dx \quad \cdots \circledast$$

Set $u = \frac{x - \bar{x}}{\sqrt{2}\sigma}$, $du = \frac{1}{\sqrt{2}\sigma}dx$ then

$$\circledast = \exp\left(\frac{\bar{x}^2}{2\sigma^2}\right) \int \exp(-u^2)\,\sqrt{2}\sigma\,du$$

$$= \exp\left(\frac{\bar{x}^2}{2\sigma^2}\right) \cdot \sqrt{2}\sigma \cdot \int \exp(-u^2)\,du$$

Since $\int e^{-u^2}du = \sqrt{\pi}$,

$$\circledast = \exp\left(\frac{\bar{x}^2}{2\sigma^2}\right) \cdot \sqrt{2}\sigma \cdot \sqrt{\pi} = \exp\left(\frac{\bar{x}^2}{2\sigma^2}\right) \cdot \sqrt{2\pi}\,\sigma$$

Now I got

$$\int \pi(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{\bar{x}^2}{2\sigma^2}\right) \int \exp\left(-\frac{x^2 - 2\bar{x}x}{2\sigma^2}\right) dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{\bar{x}^2}{2\sigma^2}\right) \cdot \exp\left(\frac{\bar{x}^2}{2\sigma^2}\right) \sqrt{2\pi}\,\sigma$$

$$= 1. \quad \square$$