Exercise 1.1.2

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Claim

If $\mathcal{F}$ is a collection of subsets of Ω with $\Omega \in \mathcal{F}$ and which is closed under complement and countable unions, then it is closed under countable intersections.

Proof

Our aim is to show that for a collection of subsets $\{A_j\}_{j \in \mathbb{N}}$ with $A_j \in \mathcal{F}\ \forall j$, we have $\bigcap_j A_j \in \mathcal{F}$.

Let us start by considering the following.

Closed Under Complement (1)
If $\mathcal{F}$ is closed under complement, then for any $A_j \in \mathcal{F}$, we have $A_j^c \in \mathcal{F}$.

Countable Unions (2)
For a collection of subsets $\{A_j\}_{j \in N}$ with $A_j \in \mathcal{F}\ \forall j$, we have that $\bigcup_j A_j \in \mathcal{F}$.

Complement of Intersection (3)
De Morgan's First Law※: $(\bigcap_j A_j)^c = \bigcup_j A_j^c$.

The proof is then straightforward.

Proof. For all $A_j \in \mathcal{F}$, we have by (1) that $A_j^c \in \mathcal{F}$. Then by (2), we have that $\bigcup_j A_j^c \in \mathcal{F}$. By (3), we have that $(\bigcap_j A_j)^c = \bigcup_j A_j^c \in \mathcal{F}$. Finally, by (1), we have that $\bigcap_j A_j \in \mathcal{F}$. □

※ De Morgan's Law for Countable Intersections

In order to justify the usage of De Morgan's Law for a countable intersection, we can use the quantified statements that define union and intersection.

$$x \in \bigcup_j A_j \iff (\exists j \in \mathbb{N}) \quad x \in A_j$$

$$x \in \bigcap_j A_j \iff (\forall j \in \mathbb{N}) \quad x \in A_j$$

Then we take the negation of the second statement.

$$\begin{aligned} x \in \left(\bigcap_j A_j\right)^c &\iff x \notin \bigcap_j A_j \\ &\iff \neg\left((\forall j \in \mathbb{N}) \quad x \in A_j\right) \\ &\iff (\exists j \in \mathbb{N}) \quad x \notin A_j \\ &\iff (\exists j \in \mathbb{N}) \quad x \in A_j^c \\ &\iff x \in \bigcup_j A_j^c \end{aligned}$$

So we may conclude that:

$$\left(\bigcap_j A_j\right)^c = \bigcup_j A_j^c$$