Langevin equation

NGUYEN Tue Tai (ID 062201848)

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Exercise 5.1.7. *Consider the Itô process satisfying*

$$\mathrm{d}X_t = -\beta X_t \,\mathrm{d}t + \alpha \,\mathrm{d}B_t \,, \qquad X_0 = x_0 \tag{1}$$

for $\alpha \in \mathbb{R}$ and $\beta > 0$. This equation is called the Langevin equation. Note that this equation can be written equivalently

$$X_t = x_0 + \alpha B_t - \beta \int_0^t X_s \,\mathrm{d}s \,.$$

Show that the solution of this equation reads

$$X_t = x_0 e^{-\beta t} + \alpha \int_0^t e^{-\beta(t-u)} \,\mathrm{d}B_u \,.$$

Consider a deterministic differential equation as follows:

$$x'(t) + \beta x(t) = f(t),$$

for some constant β and function f. We solve it by multiplying both sides by $e^{\beta t}$ and get

$$\begin{aligned} & e^{\beta t} x'(t) + \beta e^{\beta t} x(t) = f(t) e^{\beta t} \\ \Leftrightarrow \quad & \frac{\mathrm{d}}{\mathrm{d}t} \left[e^{\beta t} x \right](t) = f(t) e^{\beta t} \\ \Leftrightarrow \quad & e^{\beta t} x(t) = \int f(t) e^{\beta t} \,\mathrm{d}t + C \\ \Leftrightarrow \quad & x(t) = e^{-\beta t} \left[\int f(t) e^{\beta t} \,\mathrm{d}t + C \right]. \end{aligned}$$

Taking inspiration from this, we try an ansatz $X_t = e^{-\beta t} Z_t$ for some Itô process $(Z_t)_{t \in [0,T]}$ such that $Z_0 = x_0$. Since we have

$$\begin{aligned} \mathrm{d}\left[e^{-\beta t} \right] &= 0 \,\mathrm{d}B_t - \beta e^{-\beta t} \,\mathrm{d}t \,, \\ \mathrm{d}Z_t &= V_t \,\mathrm{d}B_t + D_t \,\mathrm{d}t \,, \end{aligned}$$

Lemma 5.1.9[1] dictates that

$$\begin{aligned}\mathrm{d}X_t = \mathrm{d}\left[e^{-\beta t} Z_t\right] &= Z_t \mathrm{d}\left[e^{-\beta t}\right] + e^{-\beta t}\,\mathrm{d}Z_t + 0\cdot V_t\,\mathrm{d}t \\ &= -\beta e^{-\beta t} Z_t\,\mathrm{d}t + e^{-\beta t}\,\mathrm{d}Z_t \\ &= -\beta X_t\,\mathrm{d}t + e^{-\beta t}\,\mathrm{d}Z_t\,.\end{aligned}$$

Comparing this with (1), we obtain Z_t:

$$e^{-\beta t}\,\mathrm{d}Z_t = \alpha\,\mathrm{d}B_t \quad\Leftrightarrow\quad \mathrm{d}Z_t = \alpha e^{\beta t}\,\mathrm{d}B_t \quad\Leftrightarrow\quad Z_t = Z_0 + \alpha\int_0^t e^{\beta u}\,\mathrm{d}B_u\,.$$

As a result, we obtain a solution of (1) by using the initial condition $Z_0 = x_0$:

$$X_t = e^{-\beta t} Z_t = x_0 e^{-\beta t} + \alpha\int_0^t e^{-\beta(t-u)}\,\mathrm{d}B_u\,.$$

The process $(X_t)_{t\in[0,T]}$ satisfying (1) is also called the *Ornstein-Uhlenbeck process.*

Now, we want to find the expectation value $\mathbb{E}(X_t)$ and the autocovariance $\mathrm{Cov}(X_t, X_s)$ of this process. Observe that

$$\int_0^T e^{2\beta u}\,\mathrm{d}u = \frac{e^{2\beta u}}{2\beta}\bigg|_{u=0}^{u=T} = \frac{e^{2\beta T}-1}{2\beta} < \infty.$$

Since the process $\left(e^{\beta u}\right)_{u\in[0,T]}$ belongs to $M^2([0,T])$, we obtain the expectation value of X_t by using **Proposition 4.2.10**,

$$\mathbb{E}(X_t) = \mathbb{E}\left(x_0 e^{-\beta t}\right) + \alpha e^{-\beta t}\mathbb{E}\left(\int_0^t e^{\beta u}\,\mathrm{d}B_u\right) = x_0 e^{-\beta t} + 0 = x_0 e^{-\beta t}.$$

Now, before finding the autocovariance $\mathrm{Cov}(X_t, X_s)$, we prove that increments of a martingale $(M_t)_{t\in[0,T]}$ are uncorrelated [2, Exercise 5.4]. Consider any times t_1, t_2, t_3, t_4 such that $t_1 \le t_2 \le t_3 \le t_4$. Then, we have by using the martingale property,

$$\mathbb{E}(M_{t_4} - M_{t_3} \mid \mathcal{F}_{t_2}) = \mathbb{E}(M_{t_4} \mid \mathcal{F}_{t_2}) - \mathbb{E}(M_{t_3} \mid \mathcal{F}_{t_2}) = M_{t_2} - M_{t_2} = \mathbf{0}.$$

Since $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$ and M_{t_1} is $\mathcal{F}_{t_1}$-measurable, M_{t_1} is also $\mathcal{F}_{t_2}$-measurable. As a result $M_{t_2} - M_{t_1}$ is $\mathcal{F}_{t_2}$-measurable and we have

$$\begin{aligned}\mathbb{E}\left[(M_{t_2} - M_{t_1})(M_{t_4} - M_{t_3})\right] &= \mathbb{E}\left\{\mathbb{E}\left[(M_{t_2} - M_{t_1})(M_{t_4} - M_{t_3}) \mid \mathcal{F}_{t_2}\right]\right\} \\ &= \mathbb{E}\left[(M_{t_2} - M_{t_1})\mathbb{E}\left(M_{t_4} - M_{t_3} \mid \mathcal{F}_{t_2}\right)\right] \qquad (\textbf{Proposition 3.1.3}, 4.) \\ &= \mathbb{E}\left[(M_{t_2} - M_{t_1})\cdot\mathbf{0}\right] = 0.\end{aligned}$$

Note that $(M_t)_{t\in[0,T]}$ defined by $M_t = \int_0^t Y_u\,\mathrm{d}B_u$ is a martingale for any adapted stochastic process $(Y_t)_{t\in[0,T]}$ belonging to $M^2([0,T])$ (**Theorem 4.3.1**). Then, using **Proposition 4.2.10** and the

[1] All the statement in bold are from the lecture notes [1].

above property of martingales, we have for any $s \le t$,

$$\mathbb{E}\left[\left(\int_0^t Y_u \,\mathrm{d}B_u\right)\left(\int_0^s Y_u \,\mathrm{d}B_u\right)\right] = \mathbb{E}\left[\left(\int_0^s Y_u \,\mathrm{d}B_u\right)^2\right] + \mathbb{E}\left[\left(\int_s^t Y_u \,\mathrm{d}B_u\right)\left(\int_0^s Y_u \,\mathrm{d}B_u\right)\right]$$
$$= \int_0^s \mathbb{E}(Y_u^2)\,\mathrm{d}u + \mathbb{E}\left[(M_t - M_s)(M_s - M_0)\right]$$
$$= \int_0^s \mathbb{E}(Y_u^2)\,\mathrm{d}u\,.$$

Now, coming back to the Ornstein-Uhlenbeck process $(X_t)_{t\in[0,T]}$, we have for any $s \le t$,

$$\begin{aligned}
\mathrm{Cov}(X_t, X_s) &= \mathbb{E}\left[(X_t - \mathbb{E}(X_t))(X_s - \mathbb{E}(X_s))\right] \\
&= \mathbb{E}\left[\left(\alpha \int_0^t e^{-\beta(t-u)}\,\mathrm{d}B_u\right)\left(\alpha \int_0^s e^{-\beta(s-u)}\,\mathrm{d}B_u\right)\right] \\
&= \alpha^2 e^{-\beta(t+s)}\mathbb{E}\left[\left(\int_0^t e^{\beta u}\,\mathrm{d}B_u\right)\left(\int_0^s e^{\beta u}\,\mathrm{d}B_u\right)\right] \\
&= \alpha^2 e^{-\beta(t+s)} \int_0^s e^{2\beta u}\,\mathrm{d}u \\
&= \alpha^2 e^{-\beta(t+s)} \frac{e^{2\beta s} - 1}{2\beta} \\
&= \frac{\alpha^2}{2\beta} e^{-\beta(t-s)} \left(1 - e^{-2\beta s}\right).
\end{aligned}$$

Furthermore, $\left(\int_0^t e^{\beta u}\,\mathrm{d}B_u\right)_{t\in[0,T]}$ is a Gaussian process by **Theorem 4.3.4**. We know that if X is a Gaussian random variable, then $aX + b$ is also a Gaussian random variable for any a, b with $a \neq 0$. So $(X_t)_{t\in[0,T]}$ is also a Gaussian process since $X_t = x_0 e^{-\beta t} + \alpha e^{-\beta t} \int_0^t e^{\beta u}\,\mathrm{d}B_u$ for any $t \in [0, T]$.

If $\alpha = \beta = 1$ and $x_0 = 0$, then we get $\mathbb{E}(X_t) = 0$ and $\mathrm{Cov}(X_t, X_s) = \frac{1}{2} e^{-(t-s)}\left(1 - e^{-2s}\right)$ for $s \le t$, which is the Gaussian process given in **Example 2.2.6**.

Digression: Langevin equation in physics

Originally, the Langevin equation was proposed as the equation of motion for a small particle in Brownian motion (see [3, Chapter 7]). Consider a particle of mass m undergoing Brownian motion in a fluid. The particle is assumed to be larger and heavier than the constituents of the fluid. In addition, the particle is subjected to a damping force proportional to the particle's velocity with the coefficient γ, and a random force $\xi(t)$ due to the thermal agitation of molecules composing the fluid. The equation of motion according to Newton's 2^{nd} law is

$$m \frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\gamma v(t) + \xi(t), \tag{2}$$

which we write in one dimension for simplicity. The random force $\xi(t)$ is also called *white noise*, which is assumed to be a Gaussian process with stationary (invariant under time shifts) and Markov properties. $\xi(t)$ also has the following averaged properties

$$\langle \xi(t) \rangle = 0, \qquad \langle \xi(t)\xi(t') \rangle = g\delta(t-t'), \tag{3}$$

where δ is the delta Dirac function and g is the measure of the noise's strength. In the notation of this course, the equations (2) and (3) can be rewritten as

$$m\,\mathrm{d}V_t = -\gamma V_t\,\mathrm{d}t + \Xi_t\,\mathrm{d}t\,, \tag{4}$$

$$\mathbb{E}(\Xi_t) = 0, \qquad \mathbb{E}(\Xi_t \Xi_{t'}) = g\delta(t-t'), \tag{5}$$

where we replace v and ξ with their capital counterparts. Comparing (4) with (1), we get $\beta = \gamma/m$ for the first term and formally for the second term,

$$\Xi_t \text{ `` = '' } m\alpha \frac{\mathrm{d}B_t}{\mathrm{d}t}.$$

Since we know that the Brownian motion (B_t) is nowhere differentiable, this expression of Ξ_t is not well-defined. However, the version (4) of the Langevin equation is widely used in physics. Therefore, a heuristic explanation (see [4]) can be made to justify the usage of Ξ_t (or $\xi(t)$) and its properties (5). From the definition of the Brownian motion, we have for any $t, \delta t > 0$,

$$(B_{t+\delta t} - B_t) \sim N(0, \delta t) \quad \Rightarrow \quad \Xi_t^{\delta t} := m\alpha \frac{B_{t+\delta t} - B_t}{\delta t} \sim N\left(0, \frac{m^2\alpha^2}{\delta t}\right).$$

It is clear that $\mathbb{E}\left(\Xi_t^{\delta t}\right) = 0$ for any $t, \delta t > 0$. Since the Brownian motion B_t is also a martingale, we can use the property that the increments are uncorrelated to compute the autocovariance $\mathbb{E}(\Xi_t^{\delta t}\Xi_{t'}^{\delta t})$. Fix the value of t. For any t' such that $t' + \delta t < t$ or $t + \delta t < t'$, we have

$$\mathbb{E}\left[(B_{t+\delta t} - B_t)(B_{t'+\delta t} - B_{t'})\right] = 0 \quad \Rightarrow \quad \mathbb{E}(\Xi_t^{\delta t}\Xi_{t'}^{\delta t}) = 0.$$

Now, consider t' such that $t' \le t \le t' + \delta t$. Then, we have $t' \le t \le t' + \delta t \le t + \delta t$ and

$$\begin{aligned}
\mathbb{E}\left[(B_{t+\delta t} - B_t)(B_{t'+\delta t} - B_{t'})\right] &= \mathbb{E}\left[(B_{t+\delta t} - B_{t'+\delta t} + B_{t'+\delta t} - B_t)(B_{t'+\delta t} - B_t + B_t - B_{t'})\right] \\
&= \mathbb{E}\left[(B_{t+\delta t} - B_{t'+\delta t})(B_{t'+\delta t} - B_t)\right] + \mathbb{E}\left[(B_{t'+\delta t} - B_t)^2\right] \\
&\quad + \mathbb{E}\left[(B_{t+\delta t} - B_{t'+\delta t})(B_t - B_{t'})\right] + \mathbb{E}\left[(B_{t'+\delta t} - B_t)(B_t - B_{t'})\right] \\
&= 0 + (t' + \delta t - t) + 0 + 0 \\
&= \delta t - (t - t'),
\end{aligned}$$

where we use the fact that $(B_{t'+\delta t} - B_t) \sim N(0, t' + \delta t - t)$. For the case $t \le t' \le t + \delta t$, since we also have $t \le t' \le t + \delta t \le t' + \delta t$, we can use the above result and interchange t and t':

$$\mathbb{E}\left[(B_{t+\delta t} - B_t)(B_{t'+\delta t} - B_{t'})\right] = \delta t + (t - t').$$

As a result, we get the autocovariance depending only on $(t - t')$,

$$\mathbb{E}(\Xi_t^{\delta t}\Xi_{t'}^{\delta t}) = \begin{cases} m^2\alpha^2 \frac{\delta t - |t-t'|}{\delta t^2}, & |t - t'| \le \delta t, \\ 0, & |t - t'| > \delta t. \end{cases}$$

In addition, we have

$$
\begin{aligned}
\int_{t-\delta t}^{t+\delta t} (\delta t - |t-t'|)\,\mathrm{d}t' &= \int_{t-\delta t}^{t} (\delta t - t + t')\,\mathrm{d}t' + \int_{t}^{t+\delta t} (\delta t + t - t')\,\mathrm{d}t' \\
&= \left(\delta t^2 - t\delta t + \frac{2t\delta t - \delta t^2}{2}\right) + \left(\delta t^2 + t\delta t - \frac{2t\delta t + \delta t^2}{2}\right) \\
&= \delta t^2.
\end{aligned}
$$

Therefore, we get

$$
\int_{-\infty}^{\infty} \mathbb{E}(\Xi_t^{\delta t}\Xi_{t'}^{\delta t})\,\mathrm{d}t' = m^2\alpha^2\frac{\delta t^2}{\delta t^2} = m^2\alpha^2.
$$

As δt approaches 0, the region, where $\mathbb{E}(\Xi_t^{\delta t}\Xi_{t'}^{\delta t})$ has nonzero values, shrinks while $\mathbb{E}(\Xi_t^{\delta t}\Xi_{t'}^{\delta t}) = m^2\alpha^2/\delta t \to \infty$ for $t = t'$. Meanwhile, the integral of $\mathbb{E}(\Xi_t^{\delta t}\Xi_{t'}^{\delta t})$ with respect to t or t' on $\mathbb{R}$ remains a constant. Hence, $\mathbb{E}(\Xi_t^{\delta t}\Xi_{t'}^{\delta t})$ formally converges to $m^2\alpha^2\delta(t-t')$ as δt goes to 0. Comparing with (5), we get the equality $\alpha^2 = g/m^2$. Consequently, we can define formally

$$
\Xi_t := \lim_{\delta t\to 0} \Xi_t^{\delta t} = \sqrt{g}\lim_{\delta t\to 0}\frac{B_{t+\delta t} - B_t}{\delta t} \quad \text{“ = ”} \quad \sqrt{g}\frac{\mathrm{d}B_t}{\mathrm{d}t},
$$

and get all the properties required. In hindsight, we can rewrite (4) in a more rigorous form:

$$
m\,\mathrm{d}V_t = -\gamma V_t\,\mathrm{d}t + \sqrt{g}\,\mathrm{d}B_t\,. \tag{6}
$$

Even though (4) is mostly used in physics, it is a mnemonic for (6), which is more mathematically precise.

References

[1] Serge Richard, Lecture Notes: Introduction to Stochastic Calculus (Fall 2023)

[2] J.-L. Arguin, A first course in stochastic calculus

[3] L. E. Reichl, A modern course in statistical physics

[4] Gillespie, Daniel T. “The Mathematics of Brownian Motion and Johnson Noise.” *American Journal of Physics* 64, no. 3 (March 1, 1996): 225–40. https://doi.org/10.1119/1.18210.