The Bounded Linear Map $X \mapsto \mathbb{E}(X \mid \mathcal{G})$ from $L^p(\Omega, \mathcal{F}, \mathbb{P})$ to $L^p(\Omega, \mathcal{G}, \mathbb{P})$

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Exercise 3.1.5. *In the framework of the previous proposition and for univariate random variables, show that the map $X \mapsto \mathbb{E}(X \mid \mathcal{G})$ is a bounded linear map from $L^p(\Omega, \mathcal{F}, \mathbb{P})$ to $L^p(\Omega, \mathcal{G}, \mathbb{P})$ with a norm smaller than or equal to 1, for any $p \geq 1$. More explicitly, show the linearity and that $\mathbb{E}\left[|\mathbb{E}(X \mid \mathcal{G})|^p\right] \leq \mathbb{E}(|X|^p)$. In the proof, use Jensen's inequality for the function $x \mapsto |x|^p$.*

For any $X^1, X^2 \in L^p(\Omega, \mathcal{F}, \mathbb{P})$ and $\alpha, \beta \in \mathbb{R}$, we have by the definition of conditional expectations, for any $D \in \mathcal{G}$,

$$\begin{aligned}
\int_D \mathbb{E}(\alpha X^1 + \beta X^2 \mid \mathcal{G}) \, \mathrm{d}\mathbb{P} &= \int_D (\alpha X^1 + \beta X^2) \, \mathrm{d}\mathbb{P} \\
&= \alpha \int_D X^1 \, \mathrm{d}\mathbb{P} + \beta \int_D X^2 \, \mathrm{d}\mathbb{P} \\
&= \alpha \int_D \mathbb{E}\left(X^1 \mid \mathcal{G}\right) \mathrm{d}\mathbb{P} + \beta \int_D \mathbb{E}\left(X^2 \mid \mathcal{G}\right) \mathrm{d}\mathbb{P} \\
&= \int_D \left(\alpha \mathbb{E}\left(X^1 \mid \mathcal{G}\right) + \beta \mathbb{E}\left(X^2 \mid \mathcal{G}\right)\right) \mathrm{d}\mathbb{P} \,.
\end{aligned}$$

Observe that $\alpha \mathbb{E}\left(X^1 \mid \mathcal{G}\right) + \beta \mathbb{E}\left(X^2 \mid \mathcal{G}\right)$ is $\mathcal{G}$-measurable. Since $\mathbb{E}(\alpha X^1 + \beta X^2 \mid \mathcal{G})$ is defined up to a set of $\mathbb{P}$-measure 0, we have the linear property

$$\mathbb{E}(\alpha X^1 + \beta X^2 \mid \mathcal{G}) = \alpha \mathbb{E}\left(X^1 \mid \mathcal{G}\right) + \beta \mathbb{E}\left(X^2 \mid \mathcal{G}\right).$$

Hence, $\mathbb{E}(\,\cdot \mid \mathcal{G})$ is a linear map from $L^p(\Omega, \mathcal{F}, \mathbb{P})$ to $L^p(\Omega, \mathcal{G}, \mathbb{P})$. Next, we define the norm

$$\|\mathbb{E}(\,\cdot \mid \mathcal{G})\| = \sup_{X \in L^p(\Omega, \mathcal{F}, \mathbb{P})} \frac{\|\mathbb{E}(X \mid \mathcal{G})\|_p}{\|X\|_p}.$$

We want to show that this norm is finite.

First of all, we will show that the function $x \mapsto |x|^p$ is convex. Let x, y be any real number. For $p = 1$, using the triangle inequality, we have for any $t \in [0, 1]$

$$|(1-t)x + ty| \leq |(1-t)x| + |ty| = (1-t)|x| + t|y|.$$

Thus, $x \mapsto |x|$ is a convex function. For any $p > 1$, we also have for any $t \in [0, 1]$,

$$|(1-t)x + ty| \le (1-t)|x| + t|y|.$$

Now, we use Hölder's inequality:

$$|a_1 b_1| + |a_2 b_2| \le \left(|a_1|^p + |a_2|^p\right)^{1/p} \left(|b_1|^q + |b_2|^q\right)^{1/q},$$

for any $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $a_1 = (1-t)^{1/p}|x|$, $b_1 = (1-t)^{1/q}$, $a_2 = t^{1/p}|y|$, and $b_2 = t^{1/q}$. Then, we have

$$\begin{aligned} |(1-t)x + ty| &\le \left|(1-t)^{1/p}|x|\right| \left|(1-t)^{1/q}\right| + \left|t^{1/p}|y|\right| \left|t^{1/q}\right| \\ &\le \left[(1-t)|x|^p + t|y|^p\right]^{1/p} \left[(1-t) + t\right]^{1/q} \\ &= \left[(1-t)|x|^p + t|y|^p\right]^{1/p}, \end{aligned}$$

or equivalently,

$$|(1-t)x + ty|^p \le (1-t)|x|^p + t|y|^p.$$

As a result, $x \mapsto |x|^p$ is convex for any $p > 1$, and therefore, it is convex for any $p \ge 1$.

Now, we use Jensen's inequality (property 7 in Proposition 3.1.3 of the lecture notes): for any univariate random variable X and any convex lower semi-continuous function $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$,

$$\varphi\big[\mathbb{E}(X \mid \mathcal{G})\big] \le \mathbb{E}\big[\varphi(X) \mid \mathcal{G}\big] \quad \text{a.s.}$$

Let $\varphi(x) = |x|^p$ for any $x \in \mathbb{R}$. Then, we have for any $X \in L^p(\Omega, \mathcal{F}, \mathbb{P})$,

$$|\mathbb{E}(X \mid \mathcal{G})|^p \le \mathbb{E}\left(|X|^p \mid \mathcal{G}\right) \quad \text{a.s.} \quad \iff \quad \mathbb{E}\left(|X|^p \mid \mathcal{G}\right) - |\mathbb{E}(X \mid \mathcal{G})|^p \ge 0 \quad \text{a.s.}$$

Using the property of expectation values that $\mathbb{E}(X) \ge 0$ for any $X \ge 0$ a.s., we have

$$0 \le \mathbb{E}\Big[\mathbb{E}(|X|^p \mid \mathcal{G}) - |\mathbb{E}(X \mid \mathcal{G})|^p\Big],$$

or equivalently, by the linearity of expectations,

$$\mathbb{E}\big[|\mathbb{E}(X \mid \mathcal{G})|^p\big] \le \mathbb{E}\big[\mathbb{E}(|X|^p \mid \mathcal{G})\big] = \mathbb{E}(|X|^p).$$

Hence, we have for any $X \in L^p(\Omega, \mathcal{F}, \mathbb{P})$,

$$\frac{\|\mathbb{E}(X \mid \mathcal{G})\|_p}{\|X\|_p} = \frac{\left\{\mathbb{E}\big[|\mathbb{E}(X \mid \mathcal{G})|^p\big]\right\}^{1/p}}{\big[\mathbb{E}(|X|^p)\big]^{1/p}} = \left\{\frac{\mathbb{E}\big[|\mathbb{E}(X \mid \mathcal{G})|^p\big]}{\mathbb{E}(|X|^p)}\right\}^{1/p} \le 1.$$

Since the right-hand side is a constant, we have

$$\|\mathbb{E}(\,\cdot \mid \mathcal{G})\| = \sup_{X \in L^p(\Omega, \mathcal{F}, \mathbb{P})} \frac{\|\mathbb{E}(X \mid \mathcal{G})\|_p}{\|X\|_p} \le 1.$$

Therefore, $\mathbb{E}(\,\cdot \mid \mathcal{G})$ is a bounded linear map from $L^p(\Omega, \mathcal{F}, \mathbb{P})$ to $L^p(\Omega, \mathcal{G}, \mathbb{P})$.

Furthermore, $\mathbb{E}(\cdot \mid \mathcal{G})$ is also a continuous linear map from $L^p(\Omega, \mathcal{F}, \mathbb{P})$ to $L^p(\Omega, \mathcal{G}, \mathbb{P})$ (see for example, Wikipedia: Bounded operator for the proof in a more general setting).