The Quadratic Variation of Itô process and a Solution to an Itô Equation

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Exercise 5.1.5. Consider the Itô process of the form $X_t = X_0 + \int_0^t V_s\, dB_s + \int_0^t D_s\, ds$. The quadratic variation of X_t is given by

$$[X]_t = \int_0^t V_s^2 ds.$$

By using the previous statement, check that this can also be expressed as

$$[X]_t = X_t^2 - X_0^2 - 2\int_0^t X_u\, dX_u.$$

Proof. By the Proposition 5.1.4, if $f : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}$ is C^1-function in t and C^2 in x, then

$$f(t,\, X_t) = f(s,\, X_s) + \int_s^t [\partial_x f](u,\, X_u)\, dX_u + \int_s^t \left\{ [\partial_t f](u,\, X_u) + \frac{1}{2}(V_u)^2 [\partial_x^2 f](u,\, X_u) \right\} du$$

for all $s \le t$. In this equation, we substitute $f(t,\, x) = x^2$. Then since $[\partial_x f](t,\, x) = 2x$, $[\partial_t f](t,\, x) = 0$, $[\partial_x^2 f](t,\, x) = 2$, we get

$$\begin{aligned}
X_t^2 &= f(t,\, X_t) \\
&= f(s,\, X_s) + \int_s^t [\partial_x f](u,\, X_u)\, dX_u + \int_s^t \left\{ [\partial_t f](u,\, X_u) + \frac{1}{2}(V_u)^2 [\partial_x^2 f](u,\, X_u) \right\} du \\
&= X_s^2 + 2\int_s^t X_u\, dX_u + \int_s^t (V_u)^2 du.
\end{aligned}$$

Thus, by taking $s = 0$, we get

$$[X]_t = \int_0^t V_s^2 ds = X_t^2 - X_0^2 - 2\int_0^t X_s\, dX_s.$$

□

Exercise 5.1.6. Consider the Itô process satisfying

$$dX_t = X_t\, dB_t + \frac{1}{2} X_t\, dt, \quad X_0 = x_0,$$

and assume that $X_t \ge 0$ for all $t \ge 0$. By applying Proposition 5.1.4 to the function $t \mapsto \ln(X_t)$, show that one solution of this equation is $X_t = x_0 e^{B_t}$.

Proof. We consider substituting $f(t,\, x) = \ln(x)$ in the equation of the Proposition 5.1.4. Since $[\partial_t f](t,\, x) = 0$, $[\partial_x f](t,\, x) = \frac{1}{x}$, $[\partial_x^2 f](t,\, x) = -\frac{1}{x^2}$, we have

$$\begin{aligned}
\ln(X_t) &= f(t,\, X_t) \\
&= f(0,\, X_0) + \int_0^t [\partial_x f](s,\, X_s)\, dX_s + \int_0^t \left\{ [\partial_t f](s,\, X_s) + \frac{1}{2}(X_s)^2 [\partial_x^2 f](s,\, X_s) \right\} ds \\
&= \ln(X_0) + \int_0^t \frac{1}{X_s}\, dX_u - \frac{1}{2}\int_0^t ds \\
&= \ln(x_0) + \int_0^t \frac{1}{X_s}\left(X_s\, dB_s + \frac{1}{2} X_s\, ds \right) - \frac{t}{2} \\
&= \ln(x_0) + \int_0^t \left(dB_s + \frac{1}{2} ds \right) - \frac{t}{2} \\
&= \ln(x_0) + B_t \quad (B_0 = 0),
\end{aligned}$$

where in the fourth equality we used the relation by which the Itô process is given. The resulting equation can easily be solved by taking the exponential of both sides and we get $X_t = x_0 e^{B_t}$.

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