The Discrete Time Stochastic Integral Is A Martingale

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Theorem 4.2.2. Let $M := (\Omega, \mathcal{F}, P, (\mathcal{F}_n)_{n\in\mathbb{N}}, (M_n)_{n\in\mathbb{N}})$ be a martingale, and let $X := (X_n)_{n\in\mathbb{N}}$ be an adapted and predictable univariate stochastic process, with $(X \cdot M)_n$ belonging to $L^1(\Omega, \mathcal{F}, P)$ for any $n \in \mathbb{N}$. Then $((X \cdot M)_n)_{n\in\mathbb{N}}$ defines a martingale with the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

Proof. We need to show that $(X \cdot M)_n$ is adapted to $\mathcal{F}_n$ for any $n \in \mathbb{N}$, and that $\mathbb{E}[(X \cdot M)_n \mid \mathcal{F}_m] = (X \cdot M)_m$ for any $m \leq n$. The first condition is reduced to the fact that $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is a filtration of $(M_n)_{n\in\mathbb{N}}$ and X is adapted to it, because of the definition of the form

$$(X \cdot M)_n = X_0 M_0 + \sum_{j=1}^{n} X_j (M_j - M_{j-1}).$$

For the second condition, we have

$$\begin{aligned}
\mathbb{E}[(X \cdot M)_n \mid \mathcal{F}_m] &= \mathbb{E}\Big[X_0 M_0 + \sum_{j=1}^{n} X_j (M_j - M_{j-1}) \mid \mathcal{F}_m\Big] \\
&= \mathbb{E}[(X \cdot M)_m \mid \mathcal{F}_m] + \mathbb{E}\Big[\sum_{j=m+1}^{n} X_j (M_j - M_{j-1}) \mid \mathcal{F}_m\Big] \\
&= (X \cdot M)_m + \sum_{j=m+1}^{n} \mathbb{E}\left[X_j (M_j - M_{j-1}) \mid \mathcal{F}_m\right].
\end{aligned}$$

Therefore, we need to show that $\sum_{j=m+1}^{n} \mathbb{E}[X_j (M_j - M_{j-1}) \mid \mathcal{F}_m] = 0$. However, for each $j \in \{m+1, \cdots, n\}$,

$$\mathbb{E}[X_j (M_j - M_{j-1}) \mid \mathcal{F}_m] = \mathbb{E}\big[\mathbb{E}[X_j (M_j - M_{j-1}) \mid \mathcal{F}_{j-1}] \mid \mathcal{F}_m\big],$$

using the Proposition 3.1.3 (5) where $\mathcal{F}_m \subset \mathcal{F}_{j-1}$. Then, since X is predictable and X_j is $\mathcal{F}_{j-1}$-measurable,

$$\mathbb{E}\big[\mathbb{E}[X_j (M_j - M_{j-1}) \mid \mathcal{F}_{j-1}] \mid \mathcal{F}_m\big] = \mathbb{E}\big[X_j \mathbb{E}[(M_j - M_{j-1}) \mid \mathcal{F}_{j-1}] \mid \mathcal{F}_m\big],$$

and lastly, since M is a martingale,

$$\mathbb{E}\big[X_j \mathbb{E}[(M_j - M_{j-1}) \mid \mathcal{F}_{j-1}] \mid \mathcal{F}_m\big] = \mathbb{E}\big[X_j (M_{j-1} - M_{j-1}) \mid \mathcal{F}_m\big] = 0.$$

Therefore, $\sum_{j=m+1}^{n} \mathbb{E}[X_j (M_j - M_{j-1}) \mid \mathcal{F}_m] = 0$ and it is shown that $(X \cdot M)_n$ is a martingale. □