Martingales

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Exercise 3.2.2. Let $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ be a filtration on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let X be a univariate random variable on this space. Set $X_t := \mathbb{E}(X|\mathcal{F}_t)$. Show that $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathcal{T}}, (X_t)_{t\in\mathcal{T}})$ is a martingale.

Answer. It suffices to show that $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$, namely $\mathbb{E}(\mathbb{E}(X|\mathcal{F}_t)|\mathcal{F}_s) = \mathbb{E}(X|\mathcal{F}_s)$ for all $s < t$. Given the property of a filtration that $\mathcal{F}_s \subset \mathcal{F}_t$ for any $s < t$, it holds a.s., by Proposition 3.1.3 - 5.

Exercise 3.2.3. Consider $\mathcal{T} = \mathbb{N}$ and a sequence $(X_n)_{n\in\mathbb{N}}$ of independent and real valued random variables satisfying $\mathbb{E}(X_n) = 0$. Set $Y_n := \sum_{j=1}^n X_j$. Show that $(Y_j)_{j\in\mathbb{N}}$ and the natural filtration define a martingale.

Answer. Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be the natural filtration generated by $(Y_n)_{n\in\mathbb{N}}$. Since $Y_n = \sum_{j=1}^n X_j$, for $m < n$, $\mathbb{E}(Y_n|\mathcal{F}_m) = \sum_{j=1}^n \mathbb{E}(X_j|\mathcal{F}_m)$, by the linearity of conditional expectation (Proposition 3.1.3 - 1). For $j \leq m$, X_j is $\mathcal{F}_m$-measurable, and therefore $\mathbb{E}(X_j|\mathcal{F}_m) = X_j$ (Proposition 3.1.3 - 2). For $m < j \leq n$, on the other hand, $\mathbb{E}(X_j|\mathcal{F}_m) = \mathbb{E}(X_j) = 0$, since X_j is independent of $\mathcal{F}_m$ (Proposition 3.1.3 - 6). Thus, $\mathbb{E}(Y_n|\mathcal{F}_m) = \sum_{j=1}^n \mathbb{E}(X_j|\mathcal{F}_m) = \sum_{j=1}^m X_j = Y_m$, which means $(\Omega, \mathcal{F}_\mathbb{N}, \mathbb{P}, (\mathcal{F}_n)_{n\in\mathbb{N}}, (Y_n)_{n\in\mathbb{N}})$ is a martingale.

Exercise 3.2.4. Show that the standard 1-dimensional Brownian motion is a martingale.

Answer. The standard 1-dimensional Brownian motion $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathcal{T}}, (B_t)_{t\in\mathcal{T}})$ satisfies:

(i) $B_0 = 0$ a.s.

(ii) For any $0 \leq s \leq t$, $B_t - B_s$, the difference of the variable at time s and at time t, is independent of $\mathcal{F}_s$.

(iii) For any $0 \leq s \leq t$, $B_t - B_s$ is a Gaussian random variable with mean 0 and variance $t - s$, namely $N(0, t-s)$.

What we need to show is $\mathbb{E}(B_t|\mathcal{F}_s) = B_s$ for $s < t$, but by the linearity, $\mathbb{E}(B_t|\mathcal{F}_s) = \mathbb{E}(B_s|\mathcal{F}_s) + \mathbb{E}(B_t - B_s|\mathcal{F}_s)$. Then the first term becomes B_s because of its $\mathcal{F}_s$-measurablity while the second term becomes the mean value 0 of the random variable $B_t - B_s$ (iii), by the property of independence (ii). Therefore, $\mathbb{E}(B_t|\mathcal{F}_s) = \mathbb{E}(B_s|\mathcal{F}_s) + \mathbb{E}(B_t - B_s|\mathcal{F}_s) = B_s + 0 = B_s$, as needed.

Exercise 3.2.6. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}, (B_t)_{t\in\mathbb{R}_+})$ be the standard 1-dimensional Brownian motion. Show that the new process defined by $X_t := B_t^2$ is a submartingale, but that the process defined by $X_t := B_t^2 - t$ is a martingale.

Answer. To utilize the property of the standard 1-dimensional Brownian motion, we decompose B_t^2 into $B_t^2 = \big(B_s + (B_t - B_s)\big)^2 = B_s^2 + 2B_s(B_t - B_s) + (B_t - B_s)^2$. When $X_t = B_t^2$, for $s < t$, using the property as we did in the previous exercise,

$$\mathbb{E}(X_t|\mathcal{F}_s) = \mathbb{E}(B_t^2|\mathcal{F}_s) = \mathbb{E}(B_s^2|\mathcal{F}_s) + 2B_s\mathbb{E}(B_t - B_s|\mathcal{F}_s) + \mathbb{E}((B_t - B_s)^2|\mathcal{F}_s) = B_s^2 + (t - s),$$

where for the last term we used another expression of variance $\text{Var}(B_t - B_s) = \mathbb{E}((B_t - B_s)^2) - \mathbb{E}(B_t - B_s)^2$ and the property of the standard 1-dimensional Brownian motion that $\mathbb{E}(B_t - B_s) = 0$, $\text{Var}(B_t - B_s) = t - s$. Since $s < t$, $\mathbb{E}(B_t^2|\mathcal{F}_s) = B_s^2 + (t - s) \geq B_s^2 = X_s$. Thus, when $X_t = B_t^2$, $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}, (X_t)_{t\in\mathbb{R}_+})$ is a submartingale.

When $X_t = B_t^2 - t$, using the above result, $\mathbb{E}(X_t|\mathcal{F}_s) = \mathbb{E}(B_t^2 - t|\mathcal{F}_s) = B_s^2 - s = X_s$ for $s < t$, which shows $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}, (X_t)_{t\in\mathbb{R}_+})$ is a martingale.