08215039O Shinya Kondo.

NO.

DATE

proof.

$$\boxed{\mathbb{E}\left[\left(\sum_{j=0}^{n-1}(B_{t_{j+1}}-B_{t_j})^2-t\right)^2\right]\to 0.}$$

Since $t=\sum_{j=0}^{n-1} t_{j+1}-t_j$, the left-hand side above can be written as

$$\mathbb{E}\left[\left(\sum_{j=0}^{n-1}\{(B_{t_{j+1}}-B_{t_j})^2-(t_{j+1}-t_j)\}\right)^2\right]$$

For simplicity, define the variables $X_j=(B_{t_{j+1}}-B_{t_j})^2-(t_{j+1}-t_j)$, $j\leq n-1$.
By writing the square as the product of two sums, we get the above is

$$=\sum_{i,j=0}^{n-1}\mathbb{E}[X_iX_j].$$

Note that the X_j's have mean zero and are independent, because the increments are. Therefore the above reduces to

$$\sum_{i=0}^{n-1}\mathbb{E}[X_i^2].$$

We now develop the square of X_i. Since the increments are Gaussian. with variance $t_{i+1}-t_i$, we get.

$$\mathbb{E}[(B_{t_{i+1}}-B_{t_i})^4]=3(t_{i+1}-t_i)^2.$$

See next page

This implies

$$\mathbb{E}[X_i^2]=3(t_{i+1}-t_i)^2-2(t_{i+1}-t_i)^2+(t_{i+1}-t_i)^2.$$

putting all this together, we finally have that.

$$\mathbb{E}\left[\left(\sum_{j=0}^{n-1}(B_{t_{j+1}}-B_{t_j})^2-t\right)^2\right]=2\sum_{i=0}^{n-1}(t_{i+1}-t_i)^2$$

We have.

$$\sum_{i=0}^{n-1}(t_{i+1}-t_i)^2\leq\max_{i\leq n-1}(t_{i+1}-t_i)\cdot\sum_{i=0}^{n-1}(t_{i+1}-t_i)=\max_{i\leq n}(t_{i+1}-t_i)\cdot t.$$

This goes to 0 as the mesh of the partition goes to 0.

LOOSE LEAF L1100 7mm×37lines maruman

08215O390 近藤慎也

$$\mathbb{E}[B_t^4] = \frac{d^4}{d\theta^4}\mathbb{E}[e^{\theta B_t}]\Big|_{\theta=0}$$

$(B_{t_{j+1}} - B_{t_j})$

$$= \frac{d^4}{d\theta^4}\left(e^{\frac{1}{2}\theta^2 t}\right)\Big|_{\theta=0}$$

$$= \frac{d^3}{d\theta^3}\left(e^{\frac{1}{2}\theta^2 t}\theta t\right)\Big|_{\theta=0}$$

$$= \frac{d^2}{d\theta^2}\left(e^{\frac{1}{2}\theta^2 t}[t+\theta^2 t^2]\right)\Big|_{\theta=0}$$

$$= \frac{d}{d\theta}\left(e^{\frac{1}{2}\theta^2 t}[3\theta t^2 + \theta^3 t^3]\right)\Big|_{\theta=0}$$

$$= \left(e^{\frac{1}{2}\theta^2 t}[3t^2 + 6\theta^2 t^3 + \theta^4 t^4]\right)\Big|_{\theta=0}$$

$$= 3t^2 \leftarrow 3(t_{j+1} - t_j)^2$$

$\boxtimes$