Some properties of the time evolution operator

SML Introduction to Stochastic Calculus

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Exercise 6.2.4

Let $(\Lambda, \mathcal{E})$ be a standard measurable space, and let $p : \mathbb{R}_+ \times \Lambda \times \mathcal{E} \to [0,1]$ be a Markov transition function associated with a homogeneous Markov process X. For $f \in M_b(\Lambda)$, $t \geq 0$ and $y \in \Lambda$, we set

$$[U_t f](y) := \int_\Lambda f(z) p(t, y, dz).$$

Note that, by definition of Markov processes associated to Markov transition functions (Definition 6.1.7), the right-hand side is a.s. equal to the conditional expectation $\mathbb{E}[f(X_t)|X_0 = y]$. We want to prove that the operator U_t has the following properties

(1.) (Contraction) $\|U_t f\|_\infty \leq \|f\|_\infty$ for any $t \geq 0, f \in M_b(\Lambda)$. In particular, U_t maps $M_b(\Lambda)$ to $M_b(\Lambda)$.

(2.) (Semigroup) $U_s U_t = U_{s+t}$ for any $s, t \geq 0$.

(3.) ($C_b(\Lambda)$ is an invariant subspace) If p has the Feller property, then $U_t f \in C_b(\Lambda)$ whenever $f \in C_b(\Lambda)$.

Let us start by proving Property 1. For $f \in M_b(\Lambda)$, let $M = \|f\|_\infty = \sup_{z \in \Lambda} |f(z)|$. Then for any $y \in \Lambda$ and $t \geq 0$, we have

$$|[U_t f](y)| = \left| \int_\Lambda f(z) p(t, y, dz) \right| \leq \int_\Lambda |f(z)| p(t, y, dz) \leq \int_\Lambda M p(t, y, dz) = M,$$

so

$$\|U_t f\|_\infty = \sup_{y \in \Lambda} |[U_t f](y)| \leq M = \|f\|_\infty.$$

For property 2, we will need to use the Chapman-Kolmogorov equation, which reads

$$p(t, y, A) = \int_\Lambda p(t - s, z, A) p(s, y, dz)$$

for $0 \leq s < t$, $y \in \Lambda$ and $A \in \mathcal{E}$. Replacing the measures of A on the left and right hand sides by integrals of the indicator function $\mathbb{1}_A$, we get

$$\int_\Lambda \mathbb{1}_A(z) p(t, y, dz) = \int_\Lambda \left(\int_\Lambda \mathbb{1}_A(w) p(t - s, z, dw) \right) p(s, y, dz).$$

By approximating from below with simple functions and using the monotone convergence theorem, we then get for any positive measurable function f

$$\int_\Lambda f(z) p(t, y, dz) = \int_\Lambda \left(\int_\Lambda f(w) p(t - s, z, dw) \right) p(s, y, dz). \tag{1}$$

By writing $f \in M_b(\Lambda)$ as a difference of two bounded positive measurable functions, we see that this in fact holds for all $f \in M_b(\Lambda)$.

From this, we get for any $f \in M_b(\Lambda), s \geq 0, t > 0$ and $y \in \Lambda$,

$$\begin{aligned}
[U_s U_t f](y) &= \int_\Lambda [U_t f](z) p(s, y, dz) \\
&= \int_\Lambda \left(\int_\Lambda f(w) p(t, z, dw) \right) p(s, y, dz) \\
&= \int_\Lambda \left(\int_\Lambda f(w) p((s+t) - s, z, dw) \right) p(s, y, dz) \\
&= \int_\Lambda f(z) p(t + s, y, dz) \\
&= [U_{s+t} f](y),
\end{aligned}$$

(here we used (1) with $s + t$ substituted for t, noting that $s \geq 0, t > 0$ implies $0 \leq s < s + t$) so $U_s U_t = U_{s+t}$ holds for $s \geq 0, t > 0$. As for the case $t = 0$, we have

$$[U_0 f](y) = \mathbb{E}[f(X_0) \mid X_0 = y] = f(y) \quad \text{a.s,}$$

so $[U_s U_0 f] = U_s f = U_{s+0} f$. We therefore have $U_s U_t = U_{s+t}$ for all $s, t \geq 0$.

Property 3 holds essentially by definition: For p to have the Feller property means that the map

$$\Lambda \ni y \mapsto \int_\Lambda f(z) p(h, y, dz) = [U_h f](y)$$

is continuous for any bounded and continuous $f : \Lambda \to \mathbb{R}$ and any $h \geq 0$. By property 1, it is also bounded, so whenever $f \in C_b(\Lambda)$, we have $U_t f \in C_b(\Lambda)$ for all $t \geq 0$.

References

[1] P. Baldi, *Stochastic calculus, an introduction through theory and exercises*, Universitext, Springer, 2017.