Explicit Calculations of the Greeks

FIRDAUS Rafi Rizqy / 062101889

Special Mathematics Lecture: Introduction to Stochastic Calculus (Fall 2023)

Proposition 7.6.1. ([1], pg. 68)

For the Black-Scholes model and for a payoff given by $h(x) \leq C(1+|x|^\alpha)$ for some C, $\alpha > 0$. Then, the function P is given by

$$P(t,x) = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(xe^{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}z})e^{-z^2/2}dz.$$

In a European call option (with the strike price K), $P(t,x)$ is given by

$$\begin{aligned} P(t,x) &= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Big(xe^{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}z} - K\Big)^+ e^{-z^2/2}dz \qquad (1) \\ &= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{d_0(T-t,x)}^{\infty} \Big(xe^{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}z} - K\Big) e^{-z^2/2}dz. \end{aligned}$$

Since the integrand in **Equation 1** vanishes if $z \leq d_0(T-t,x)$ with

$$d_0(t,x) := \frac{1}{\sigma\sqrt{t}}\big(-\ln(x/K) - (r-\sigma^2/2)t\big).$$

Based on ([1], pg. 67), if we set

$$\Phi(y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-z^2/2}dz, \ \ d_1(t,x) = -d_0(t,x) + \sigma\sqrt{t}, \ \ d_2(t,x) = -d_0(t,x).$$

Then, one would obtain

$$P(t,x) = x\Phi(d_1(T-t,x)) - Ke^{-r(T-t)}\Phi(d_2(T-t,x)). \qquad (2)$$

Moreover, one can also obtain

$$H_t^1 = [\partial_x P](t,S_t) = \Phi(d_1(T-t,S_t)). \qquad (3)$$

Lemma 1

If we set $\phi := \Phi'$, the relation between the density function ϕ at d_1 and d_2 is given by

$$x\phi(d_1(t,x)) = Ke^{-rt}\phi(d_2(t,x)). \tag{4}$$

Proof:
First, let us consider $d_2^2 - d_1^2$

$$\begin{aligned}
d_2^2 - d_1^2 &= (d_2 - d_1)(d_2 + d_1) \\
&= \left(-\sigma\sqrt{t}\right)\left(2d_1 - \sigma\sqrt{t}\right) \\
&= \left(-\sigma\sqrt{t}\right)\left(2\left(\frac{1}{\sigma\sqrt{t}}\left(\ln(x/K) + (r + \sigma^2/2)t\right)\right) - \sigma\sqrt{t}\right) \\
&= -2\ln(x/K) - 2rt.
\end{aligned}$$

From the definition of the density function $\phi(y) = \frac{1}{\sqrt{2\pi}}e^{-y^2/2}$, one has

$$\begin{aligned}
\frac{\phi(d_2)}{\phi(d_1)} &= e^{(d_1^2 - d_2^2)/2} \\
&= \frac{xe^{rt}}{K} \\
\implies x\phi(d_1(t,x)) &= Ke^{-rt}\phi(d_2(t,x)).
\end{aligned}$$

□

Exercise 7.6.2. ([1], pg. 68)

Check these expressions, based on the definitions given in the previous section.

$$\begin{aligned}
\Gamma(t,x) &= \frac{\phi(d_1(T-t,x))}{x\sigma\sqrt{T-t}}, \\
\Theta(t,x) &= -\frac{x\phi(d_1(T-t,x))\sigma}{2\sqrt{T-t}} - rxe^{-r(T-t)}\phi(d_2(T-t,x)), \\
\rho(t,x) &= x(T-t)e^{-r(T-t)}\Phi(d_2(T-t,x)), \\
\text{Vega}(t,x) &= x\sqrt{T-t}\phi(d_1(T-t,x)).
\end{aligned}$$

Let us calculate $\Gamma(t,x), \Theta(t,x), \rho(t,x)$, and $\text{Vega}(t,x)$ by using **Equation (2), (3)**, and **Lemma 1**

$$
\begin{aligned}
\Gamma(t,x) &= [\partial_x^2 P](t,x) \\
&= \partial_x(\Phi(d_1(T-t,x))) \\
&= \frac{\partial(d_1(T-t,x))}{\partial x}\phi(d_1(T-t,x)) \\
&= -\frac{\partial(d_0(T-t,x))}{\partial x}\phi(d_1(T-t,x)) \\
&= \frac{\phi(d_1(T-t,x))}{x\sigma\sqrt{T-t}}, \\
\Theta(t,x) &= [\partial_t P](t,x) \\
&= -\frac{\partial(d_1(T-t,x))}{\partial(T-t)}x\phi(d_1(T-t,x)) - rKe^{-r(T-t)}\Phi(d_2(T-t,x)) \\
&\quad + Ke^{-r(T-t)}\frac{\partial(d_2(T-t,x))}{\partial(T-t)}\phi(d_2(T-t,x)) \\
&= -\frac{\partial(d_1(T-t,x))}{\partial(T-t)}x\phi(d_1(T-t,x)) - rKe^{-r(T-t)}\Phi(d_2(T-t,x)) \\
&\quad + x\left(\frac{\partial(d_1(T-t,x))}{\partial(T-t)} - \frac{\sigma}{2\sqrt{T-t}}\right)\phi(d_1(T-t,x)) \quad (\text{by } \textbf{Lemma 1}) \\
&= -\frac{x\phi(d_1(T-t,x))\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}\Phi(d_2(T-t,x)), \\
\rho(t,x) &= [\partial_r P](t,x) \\
&= \frac{\partial(d_1(T-t,x))}{\partial r}x\phi(d_1(T-t,x)) + K(T-t)e^{-r(T-t)}\Phi(d_2(T-t,x)) \\
&\quad - Ke^{-r(T-t)}\frac{\partial(d_2(T-t,x))}{\partial r}\phi(d_2(T-t,x)) \\
&= -\frac{\partial(d_0(T-t,x))}{\partial r}x\phi(d_1(T-t,x)) + K(T-t)e^{-r(T-t)}\Phi(d_2(T-t,x)) \\
&\quad + \frac{\partial(d_0(T-t,x))}{\partial r}x\phi(d_1(T-t,x)) \quad (\text{by } \textbf{Lemma 1}) \\
&= K(T-t)e^{-r(T-t)}\Phi(d_2(T-t,x)), \\
\text{Vega}(t,x) &= [\partial_\sigma P](t,x) \\
&= \frac{\partial(d_1(T-t,x))}{\partial\sigma}x\phi(d_1(T-t,x)) - Ke^{-r(T-t)}\frac{\partial(d_2(T-t,x))}{\partial\sigma}\phi(d_2(T-t,x)) \\
&= \frac{\partial(d_1(T-t,x))}{\partial r}x\phi(d_1(T-t,x)) - \left(\frac{\partial(d_1(T-t,x))}{\partial r} - \sqrt{T-t}\right)x\phi(d_1(T-t,x)) \quad (\text{by } \textbf{Lemma 1}) \\
&= x\sqrt{T-t}\phi(d_1(T-t,x)).
\end{aligned}
$$

References

[1] Serge Richard. *Special Mathematics Lecture: Introduction to Stochastic Calculus*. 2023.