# Markov's Inequality Proof

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## 1 Exercise

Let $(\Omega, \boldsymbol{F}, \boldsymbol{P})$ be a probability space, and let $\boldsymbol{X} : \Omega \to \boldsymbol{R}$ be a non-negative random variable. Then for any $\alpha > 0$ the following inequality holds:

$$\boldsymbol{P}(\boldsymbol{X} > \alpha) \leq \frac{1}{a}\boldsymbol{E}(\boldsymbol{X})$$

Proof: From Definition 1.2.1:

$$\boldsymbol{E}(\boldsymbol{X}) := \int_{\Lambda} \boldsymbol{x}\mu(\boldsymbol{x})d\boldsymbol{x}$$

As $\boldsymbol{X}$ is non-negative variable, $\Lambda$ could be rewritten as $[0,\infty)$

$$\boldsymbol{E}(\boldsymbol{X}) := \int_0^{\infty} \boldsymbol{x}\mu(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$

As $\alpha > 0$, we can write that $\int_0^{\infty} \boldsymbol{x}\mu(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \geq \int_{\alpha}^{\infty} \boldsymbol{x}\mu(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$ (1)

Looking at the left side of the inequality, we are particularly interested in $\boldsymbol{X} > \alpha$,

It follows from (1) that

$$\boldsymbol{E}(\boldsymbol{X}) \geq \int_{\alpha}^{\infty} \boldsymbol{x}\mu(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \geq \int_{\alpha}^{\infty} \alpha\mu(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \alpha \int_{\alpha}^{\infty} \mu(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \ (2)$$

At the same time

$$\alpha \int_{\alpha}^{\infty} \mu(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \alpha\boldsymbol{P}(\boldsymbol{X} > \alpha)$$

Rewriting (2):

$$\Leftrightarrow \boldsymbol{E}(\boldsymbol{X}) \geq \alpha\boldsymbol{P}(\boldsymbol{X} > \alpha)$$
$$\Leftrightarrow \quad \frac{1}{\alpha}\boldsymbol{E}(\boldsymbol{X}) \geq \boldsymbol{P}(\boldsymbol{X} > \alpha)$$