SML: Introduction to Stochastic Calculus
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Moment generating function for the univariate Gaussian random variable

We start by evaluating the moment generating function for a univariate Gaussian random variable $N(\bar{x}, \sigma^2)$

By definition, the mgf is given by $a \mapsto \mathbb{E}(e^{ax})$.

Therefore we have,

$$M_X(a) = \mathbb{E}(e^{ax})$$

$$= \int_\Lambda e^{ax} \mu_X(dx)$$

$$= \int_{-\infty}^{\infty} e^{ax} \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-1}{2\sigma^2}(x-\bar{x})^2\right) dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left(ax - \frac{(x-\bar{x})^2}{2\sigma^2}\right) dx$$

Substituting $u = \frac{x-\bar{x}}{\sqrt{2}\sigma}$ $\left(du = \frac{1}{\sqrt{2}\sigma} dx\right)$

$$= \frac{\sqrt{2}\sigma}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left((\sqrt{2}\sigma u + \bar{x})a - u^2\right) du$$

$$= \frac{\exp(\bar{x}a)}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-(u^2 - \sqrt{2}\sigma ua)\right) du$$

$$= \frac{\exp(\bar{x}a)}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-\left(u - \frac{\sigma a}{\sqrt{2}}\right)^2 + \frac{1}{2}\sigma^2 a^2\right) du$$

$$= \frac{\exp\left(\bar{x}a + \frac{1}{2}\sigma^2 a^2\right)}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-\left(u - \frac{\sigma a}{\sqrt{2}}\right)^2\right) du$$

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$$= \frac{\exp\left(\bar{x}a + \frac{1}{2}\sigma^2 a^2\right)}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-\left(u - \frac{\sigma a}{\sqrt{2}}\right)^2\right) du$$

substitute

$v = u - \frac{\sigma a}{\sqrt{2}}$

$$= \frac{\exp\left(\bar{x}a + \frac{1}{2}\sigma^2 a^2\right)}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-v^2\right) du$$

$$= \frac{\exp\left(\bar{x}a + \frac{1}{2}\sigma^2 a^2\right)}{\sqrt{\pi}} \sqrt{\pi}$$

$$= \exp\left(\bar{x}a + \frac{1}{2}\sigma^2 a^2\right) \blacksquare$$