BROWNIAN MARTINGALES

SML INTRODUCTION TO STOCHASTIC CALCULUS

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In this report, we shall give some fundamental examples of martingales related to Brownian motions. Let us first recall the definition of martingales, supermartingales and submartingales:

Definition 1 (martingale, supermartingale, submartingale)**.** *For $\mathcal{T} \subset \mathbb{R}_+$, a real value stochastic process $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathcal{T}}, (M_t)_{t\in\mathcal{T}})$ satisfying $M_t \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ for any $t \in \mathcal{T}$ is a* ***martingale*** *if $\mathbb{E}(M_t|\mathcal{F}_s) = M_s$ for all $s \leq t$. It is a* ***supermartingale*** *if $\mathbb{E}(M_t|\mathcal{F}_s) \leq M_s$ or a* ***submartingale*** *if $\mathbb{E}(M_t|\mathcal{F}_s) \geq M_s$.*

We intend to prove several statements by following the above definition.

Proposition 2. *The standard 1-dimensional Brownian motion is a martingale.*

Proof. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, (B_t)_{t\in\mathbb{R}_+})$ be the standard 1-dimensional Brownian motion. We see that for all $t \in \mathbb{R}_+$, $B_t \in L^2(\Omega, \mathcal{F}, \mathbb{P}) \subset L^1(\Omega, \mathcal{F}, \mathbb{P})$ as

$$\mathbb{E}(B_t^2) = \mathrm{Var}(B_t) + \mathbb{E}(B_t)^2 = t < \infty.$$

Now note that for all $t, s \in \mathbb{R}_+$ such that $t \geq s$,

$$\begin{aligned}\mathbb{E}(B_t|\mathcal{F}_s) &= \mathbb{E}(B_s + B_t - B_s|\mathcal{F}_s)\\ &= \mathbb{E}(B_s|\mathcal{F}_s) + \mathbb{E}(B_t - B_s|\mathcal{F}_s)\\ &= B_s + \mathbb{E}(B_t - B_s)\\ &= B_s,\end{aligned}$$

which implies that $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, (B_t)_{t\in\mathbb{R}_+})$ is a martingale. □

Proposition 3. *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, (B_t)_{t\in\mathbb{R}_+})$ be the standard 1-dimensional Brownian motion. Let the geometric Brownian motion be defined by $S_t := S_0 \exp(\sigma B_t + \mu t)$, with $\sigma > 0, \mu \in \mathbb{R}$, and $S_0 \in \mathbb{R}$ an arbitrary initial value. Then this process is a martingale if and only if $\mu = -\frac{1}{2}\sigma^2$.*

To prove this statement, we shall make use of the lemma given below, which is a slightly stronger result compared with **Proposition 3.1.3** 4. given in the lecture notes.

Lemma 4. *Let X be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a standard measurable space $(\Lambda, \mathcal{E})$ and further assume that $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{G}$ be a σ-subalgebra of $\mathcal{F}$. If W is an univariate $\mathcal{G}$-measurable random variable such that $WX \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, then $\mathbb{E}(WX|\mathcal{G}) = W\mathbb{E}(X|\mathcal{G})$ a.s..*

Proof. We may assume W is nonnegative, otherwise, we may write $W = W_+ - W_-$ and apply linearity, where $W_+ = \max\{W, 0\}, W_- = \max\{-W, 0\}$. Then there exists a sequence $\{\phi_n\}$ of increasing nonnegative simple functions converging to W pointwise. Then for all $n \in \mathbb{N}$, we know that $\phi_n X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$.
For the case in which X is nonnegative, for any $D \in \mathcal{G}$, by the monotone convergence theorem, we have

$$\int_D \mathbb{E}(WX|\mathcal{G})d\mathbb{P} = \int_D WXd\mathbb{P} = \lim_{n\to\infty}\int_D \phi_n Xd\mathbb{P} = \lim_{n\to\infty}\int_D \phi_n\mathbb{E}(X|\mathcal{G})d\mathbb{P} = \int_D W\mathbb{E}(X|\mathcal{G})d\mathbb{P},$$

implying $\mathbb{E}(WX|\mathcal{G}) = W\mathbb{E}(X|\mathcal{G})$ a.s., where the third equality comes from the fact that ϕ_n is a simple function.
For the general case in which X is not always nonnegative, from Jensen's inequality, we have

$$\mathbb{E}(|X|\,|\mathcal{G}) \geq |\mathbb{E}(X|\mathcal{G})|.$$

Hence

$$|\phi_n\mathbb{E}(X|\mathcal{G})| \leq \phi_n\mathbb{E}(|X|\,|\mathcal{G}) \leq W\mathbb{E}(|X|\,|\mathcal{G}).$$

From our previous result, we know that $\mathbb{E}(W\,|X|\,|\mathcal{G}) = W\mathbb{E}(|X|\,|\mathcal{G})$ a.s., thus $W\mathbb{E}(|X|\,|\mathcal{G}) \in L^1(\Omega, \mathcal{G}, \mathbb{P})$.
By dominated convergence theorem, we obtain

$$\int_D \mathbb{E}(WX|\mathcal{G})d\mathbb{P} = \int_D WXd\mathbb{P} = \lim_{n\to\infty}\int_D \phi_n Xd\mathbb{P} = \lim_{n\to\infty}\int_D \phi_n\mathbb{E}(X|\mathcal{G})d\mathbb{P} = \int_D W\mathbb{E}(X|\mathcal{G})d\mathbb{P},$$

implying $\mathbb{E}(WX|\mathcal{G}) = W\mathbb{E}(X|\mathcal{G})$ a.s., as desired. □

Now we are ready to prove **Proposition 3**.

*Proof of **Proposition 3**.* First, we know that for all $t \in \mathbb{R}_+$, $S_t \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ as

$$\mathbb{E}(S_t) = S_0 e^{\mu t}\mathbb{E}(e^{\sigma B_t}) = S_0 e^{\mu t + \frac{1}{2}\sigma^2 t} < \infty.$$

(For the last equality, we make use of the moment generating function of a Gaussian random variable.)

For all $t, s \in \mathbb{R}_+$ such that $t \geq s$,

$$\begin{aligned}
\mathbb{E}(S_t|\mathcal{F}_s) &= S_0 e^{\mu t}\mathbb{E}(e^{\sigma(B_s + B_t - B_s)}|\mathcal{F}_s) \\
&= S_0 e^{\mu t + \sigma B_s}\mathbb{E}(e^{\sigma(B_t - B_s)}|\mathcal{F}_s) \quad (\text{by } \textbf{Lemma } 4) \\
&= S_0 e^{\mu t + \sigma B_s}\mathbb{E}(e^{\sigma(B_t - B_s)}) \\
&= S_0 e^{\sigma B_s + \mu s + \mu(t-s) + \frac{\sigma^2}{2}(t-s)} \\
&= S_s e^{(\mu + \frac{\sigma^2}{2})(t-s)}
\end{aligned}$$

Therefore, we see that

$$\text{for all } t \geq s,\ \mathbb{E}(S_t|\mathcal{F}_s) = S_s \iff \mu + \frac{\sigma^2}{2} = 0 \iff \mu = -\frac{\sigma^2}{2},$$

as desired. □

Proposition 5. *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, (B_t)_{t\in\mathbb{R}_+})$ be the standard 1-dimensional Brownian motion. The process defined by $X_t := B_t^2$ is a submartingale, and the process defined by $Y_t := B_t^2 - t$ is a martingale.*

Proof. We know $X_t, Y_t \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ from our proof of **Proposition** 2.

For all $t, s \in \mathbb{R}_+$ such that $t \geq s$, from Jensen's inequality, we see that

$$\mathbb{E}(X_t|\mathcal{F}_s) = \mathbb{E}(B_t^2|\mathcal{F}_s) \geq \mathbb{E}(B_t|\mathcal{F}_s)^2 = B_s^2 = X_s^2,$$

implying the process $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, (X_t)_{t\in\mathbb{R}_+})$ is a submartingale.

For all $t, s \in \mathbb{R}_+$ such that $t \geq s$,

$$\begin{aligned}
\mathbb{E}(Y_t|\mathcal{F}_s) &= \mathbb{E}(B_t^2 - t|\mathcal{F}_s) \\
&= \mathbb{E}((B_s + B_t - B_s)^2|\mathcal{F}_s) - t \\
&= \mathbb{E}(B_s^2|\mathcal{F}_s) + 2\mathbb{E}(B_s(B_t - B_s)|\mathcal{F}_s) + \mathbb{E}((B_t - B_s)^2|\mathcal{F}_s) - t \\
&= B_s^2 + 2B_s\mathbb{E}(B_t - B_s) + \mathbb{E}((B_t - B_s)^2) - t \quad (\text{by } \textbf{Lemma } 4) \\
&= B_s^2 + t - s - t \\
&= B_s^2 - s \\
&= Y_s.
\end{aligned}$$

Therefore, $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, (Y_t)_{t\in\mathbb{R}_+})$ is a martingale, as desired. □