Exercise 1.1.3 *If Ω contains N elements, how many elements does its power set contain ? Provide an easy and understandable description of this power set.*

Let us begin with an important observation of Pascal's pyramid: The sum of the elements of one row is twice the sum of the elements of the prior row.

N \ n	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

N \ n	0	1	2	3	4	5	6
0	${}^0C_0$						
1	${}^1C_0$	${}^1C_1$					
2	${}^2C_0$	${}^2C_1$	${}^2C_2$				
3	${}^3C_0$	${}^3C_1$	${}^3C_2$	${}^3C_3$			
4	${}^4C_0$	${}^4C_1$	${}^4C_2$	${}^4C_3$	${}^4C_4$		
5	${}^5C_0$	${}^5C_1$	${}^5C_2$	${}^5C_3$	${}^5C_4$	${}^5C_5$	
6	${}^6C_0$	${}^6C_1$	${}^6C_2$	${}^6C_3$	${}^6C_4$	${}^6C_5$	${}^6C_6$

$$\sum_{n=0}^{N} {}^{N}C_n = 2\sum_{n=0}^{N-1} {}^{N-1}C_n \tag{1}$$

Proof. It is simple to see 1 is true for $N <= 3$, And for $N > 3$:

$$\begin{aligned}
N\text{th row of Pascal's triangle} &= \sum_{n=0}^{N} {}^{N}C_n \\
&= {}^{N}C_0 + {}^{N}C_0 + {}^{N}C_1 + ... + {}^{N}C_{N-1} + {}^{N}C_N \\
&= {}^{N-1}C_0 + ({}^{N-1}C_0 + {}^{N-1}C_1) + ... + ({}^{N-1}C_{N-2} + {}^{N-1}C_{N-1}) + {}^{N-1}C_{N-1} \\
&= 2({}^{N-1}C_0 + {}^{N-1}C_1 + ... + {}^{N-1}C_{N-2} + {}^{N-1}C_{N-1}) \\
&= 2\sum_{n=0}^{N-1} {}^{N-1}C_n = 2 \cdot (N-1)\text{th row of Pascal's triangle}
\end{aligned}$$

□

Definition of Power set

A set Ω which contains finitely many elements N will generate a power set $P(\Omega)$ which contains 2^N elements, with each element being a set that is a distinct combination of elements in Ω including Ω and the null set.

Size of the Power Set

Imagine an N - bit binary register and assign a single bit to each element in Ω. Each switch has two states - either on or off - akin to having or not having the element in a particular combination of elements for our power set. In this way we could represent the null set as the register having all of its bits switched off and the whole set, Ω as being the entire register switched on. The set of all sets then will have as many elements as there are combinations in our register given by

$$\#\text{ Elements in power set} = \#\text{ combinations} = \#\text{ of states}^{\#\text{ of bits}} = 2^N$$

Another method of reaching this conclusion is by summing up the elements in the N-th row (starting with the 0-th row at the top) of Pascal's Pyramid, which itself encodes the coefficients associated with each successive term in the expression

$$\sum_{n=0}^{N} {}^{N}C_n = 2^N \tag{2}$$

Proof. (by Induction)

Base Cases

$N = 0 : \sum_{n=0}^{0} {}^{0}C_n = {}^{0}C_0 = 1 = 2^0$

$N = 1 : \sum_{n=0}^{1} {}^{1}C_n = {}^{1}C_0 + {}^{1}C_1 = 1 + 1 = 2^1$

Assumption Step

$k \in \mathbb{Z},\ k > 1$

$N = k - 1 : \text{Assume } \sum n = 0^{N-1}({}^{N-1}C_n) = 2^{N-1}$

Induction Step

$N = k : \sum_{n=0}^{N} {}^{N}C_n \overset{\text{by1}}{=} \sum_{n=0}^{N-1} {}^{N-1}C_n = 2(2^{N-1}) = 2^N$ □

This approach has the added benefit of giving us the number of sets of size n that are to be found in the power set, allowing for a sanity check.

Example : $N = 4$

$$\Omega = \{1, 2, 3, 4\}$$

$P(\Omega)$ contains:

1 set of size 0 (which must be the null set),
4 sets of size 1 (the singleton sets),
6 sets of size 2, {1,2}, {1,3}, {1,4}, {2,3}, {2,4}, {3,4}
4 sets of size 3, {1,2,3}, {1,2,4}, {1,3,4}, {2,3,4}
1 set of size 4 (which is Ω itself)

$$\begin{aligned} P(\Omega) = &\{\{\emptyset\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \\ &\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\} \end{aligned}$$

$P(\Omega)$ has a size of $2^4 = 16$.
Alternatively,

$$\sum_{n=0}^{4} {}^{N}C_n = {}^{4}C_0 + {}^{4}C_1 + {}^{4}C_2 + {}^{4}C_3 + {}^{4}C_4 = 1 + 4 + 6 + 4 + 1 = 16$$

Arrives at the same conclusion.