Reminder VIII

standard 1D Brownian motion

- Aim : define $\int_a^b X_t \, dB_t$
  adapted univariate stochastic process

- Discrete time version : $(X_n)_{n\in\mathbb{N}}$ predictable

$$(X\cdot M)_n := X_0 M_0 + \sum_{j=1}^{n} X_j (M_j - M_{j-1})$$

martingale transform if M a martingale

$((X\cdot M)_n)_{n\in\mathbb{N}}$ is a martingale if $(M_n)_{n\in\mathbb{N}}$ is a martingale.

- Continuous time version = Itô integral

1° $M^p_{loc}([a,b])$, $M^p([a,b])$ progressively measurable + univariate + regularity

2° Elementary processes ("constant" on time intervals)

3° $$\int_a^b X_t \, dB_t = \sum_{j=0}^{n-1} X_j (B_{t_{j+1}} - B_{t_j})$$

Properties, if $X \in M^2([a,b])$, 0 expectation + isometry
variance

4° For general $X \in M^2_{(loc)}([a,b])$

elementary processes

$$\int_a^b X_t \, dB_t = \lim_{n\to\infty} \int_a^b X_{n,t} \, dB_t$$

limit in a suitable sense

- Set $Y_t := \int_0^t X_u \, dB_u$ for $t \in [0,T]$.
  adapted to the filtration

If $X \in M^2([0,T])$, then $(Y_t)_{t\in[0,T]}$ a martingale with 0 expectation and $\mathbb{E}(Y_t^2) = \int_0^t \mathbb{E}(X_u^2) \, du < \infty$

variance

名古屋大学大学院多元数理科学研究科