

Exercise 1 Compute the following limits:

$$\text{a) } \lim_{h \rightarrow 0} \frac{\sqrt{1+h} - 1}{h}; \quad \text{b) } \lim_{x \rightarrow -1} \frac{x^2 - x - 2}{|x + 1|} \quad \text{c) } \lim_{x \rightarrow 0} \frac{\sin^2(x)}{x} \quad \text{d) } \lim_{x \rightarrow \infty} \frac{(e^x - e^{-x})^2}{e^{2x} + e^{-2x}}:$$

Exercise 2 Compute the derivative of the following functions:

1. $f : \mathbb{R} \rightarrow \mathbb{R}; f(x) = e^{\frac{1}{2}x} \sin(2x),$
2. $g : \mathbb{R} \rightarrow \mathbb{R}; g(x) = \sin^2(x^2),$
3. $h : \mathbb{R} \rightarrow \mathbb{R}; h(x) = \frac{x-1}{x^2+1}.$

Exercise 3 Determine the following limit $\lim_{x \rightarrow 1} \frac{x \ln(x) - x + 1}{x \ln(x) - \ln(x)}.$

Exercise 4 Prove the following statement: Let $f : [a; b] \rightarrow \mathbb{R}$ be continuous on $[a; b]$ and differentiable on $(a; b)$. If $f'(x) \geq 0$ for any $x \in (a; b)$, then the function is increasing on $[a; b]$, and strictly increasing if $f'(x) > 0$.

Exercise 5 Consider the function $\tanh : \mathbb{R} \rightarrow \mathbb{R}$ given by $\tanh(x) := \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. Determine the range of this function, and show that it is invertible on its range. Compute the derivative of its inverse.

Exercise 6 Consider the curve in \mathbb{R}^2 defined by the relation $F(x; y) = 0$ for

$$F(x; y) = 3x^3y - y^4 + 5x^2 + 5:$$

1. Show that the point $(1; 2)$ belong to the curve, and find the slope of the tangent at the point.
2. Show that the points $(0; -\sqrt[4]{5})$ and $(0; \sqrt[4]{5})$ belong to the curve and determine the equation of the tangent to the curve at these points.
3. For the 2 points $(0; -\sqrt[4]{5})$ and $(0; \sqrt[4]{5})$, indicate the position of the curve with respect to the tangent.

Exercise 1 4 pts

$$a) \lim_{h \rightarrow 0} \frac{\sqrt{1+h} - 1}{h} = \lim_{h \rightarrow 0} \frac{(1+h) - 1}{h(\sqrt{1+h} + 1)} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{1+h} + 1} = \underline{\underline{\frac{1}{2}}}$$

$$b) \lim_{x \rightarrow -1} \frac{x^2 - x - 2}{|x+1|} = \lim_{x \rightarrow -1} \frac{(x+1)(x-2)}{-(x+1)} = -(-1-2) = \underline{\underline{3}}$$

$$c) \lim_{x \rightarrow 0} \frac{\sin^2(x)}{x} = \lim_{x \rightarrow 0} \frac{\sin(x)}{x} \sin(x) = 1 \cdot 0 = \underline{\underline{0}}$$

$$d) \lim_{x \rightarrow \infty} \frac{(e^x + e^{-x})^2}{e^{2x} + e^{-2x}} = \lim_{x \rightarrow \infty} \frac{e^{2x} + 2 + e^{-2x}}{e^{2x} + e^{-2x}}$$

$$= \lim_{x \rightarrow \infty} \frac{e^{2x}(1 + 2e^{-2x} + e^{-4x})}{e^{2x}(1 + e^{-4x})} = \underline{\underline{1}}$$

Exercise 2 6 pts

$$1. f'(x) = \frac{1}{2} e^{\frac{1}{2}x} \sin(2x) + e^{\frac{1}{2}x} \cos(2x) \cdot 2$$

$$= \underline{\underline{e^{\frac{1}{2}x} \left(\frac{1}{2} \sin(2x) + 2 \cos(2x) \right)}}$$

$$2. g'(x) = 2 \sin(x^2) \cos(x^2) \cdot 2x = \underline{\underline{4x \sin(x^2) \cos(x^2)}}$$

$$3. h'(x) = \frac{x^2 + 1 - 2x(x-1)}{(x^2 + 1)^2} = \underline{\underline{\frac{-x^2 + 2x + 1}{(x^2 + 1)^2}}}$$

Exercise 3 3 pts

All conditions for applying L'Hospital's rule are

satisfied (since $\ln(1) = 0$), then one has

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x \ln(x) - x + 1}{x \ln(x) - \ln(x)} & \stackrel{\text{H\^opital}}{=} \lim_{x \rightarrow 1} \frac{\ln(x) + x \frac{1}{x} - 1}{\ln(x) + x \frac{1}{x} - \frac{1}{x}} \\ & = \lim_{x \rightarrow 1} \frac{\ln(x)}{\ln(x) + 1 - \frac{1}{x}} \stackrel{\text{H\^opital}}{=} \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} = \underline{\underline{\frac{1}{2}}} \end{aligned}$$

Exercise 4 3 pts

Choose any $\alpha, \beta \in [a, b]$ with $\alpha < \beta$. Then

$f: [\alpha, \beta] \rightarrow \mathbb{R}$ (restriction of f on $[\alpha, \beta]$) is

continuous and differentiable on (α, β) . We can

apply the mean value thm: $\exists \eta \in (\alpha, \beta)$ s.t.

$$\frac{f(\beta) - f(\alpha)}{\beta - \alpha} = f'(\eta) \Leftrightarrow f(\beta) = f(\alpha) + f'(\eta) \underbrace{(\beta - \alpha)}_{> 0}.$$

If $f'(\eta) \geq 0$, then $f(\beta) \geq f(\alpha)$. If $f'(\eta) > 0$, then

$f(\beta) > f(\alpha)$. Since α, β are arbitrary, f is increasing or strictly increasing on $[a, b]$.

Exercise 5

3, 5

One has $\lim_{x \rightarrow -\infty} \tanh(x) = \lim_{x \rightarrow -\infty} \frac{e^{-x}(e^{2x}-1)}{e^{-x}(e^{2x}+1)} = -1,$

and $\lim_{x \rightarrow \infty} \tanh(x) = \lim_{x \rightarrow \infty} \frac{e^x(1-e^{-2x})}{e^x(1+e^{-2x})} = 1.$

For the derivative of \tanh , one has

$$\tanh' = \frac{\cosh^2 - \sinh^2}{\cosh^2} \stackrel{(*)}{=} \frac{1}{\cosh^2} > 0. \text{ Thus, the}$$

function is strictly increasing, with range $(-1, 1).$

$\Rightarrow \tanh$ has an inverse, defined on $(-1, 1),$

denoted by $\operatorname{arctanh}$. Observe from $(*)$ that

$$\tanh'(x) = 1 - \tanh^2(x). \text{ Thus, for } y \in (-1, 1):$$

$$\begin{aligned} \operatorname{arctanh}(y)' &= \frac{1}{\tanh'(\operatorname{arctanh}(y))} = \frac{1}{1 - \tanh(\operatorname{arctanh}(y))^2} \\ &= \frac{1}{1 - y^2}. \end{aligned}$$

Exercise 6

$$F(x, y) = 3x^3 y - y^4 + 5x^2 + 5$$

1. $F(1, 2) = 3 \cdot 1 \cdot 2 - 16 + 5 + 5 = 0 \Rightarrow (1, 2)$ belongs

to the curve. Suppose that locally, one has

$y = y(x)$ which satisfies $F(x, y(x)) = 0$ for

x belonging to a small interval of \mathbb{R} .

Then $\frac{d}{dx} F(x, y(x)) = 0$

$$\Leftrightarrow 9x^2 y(x) + 3x^3 y'(x) - 4y^3(x) y'(x) + 10x = 0$$

$$\Leftrightarrow y'(x) (3x^2 - 4y^3(x)) = -9x^2 y(x) - 10x$$

$$\Leftrightarrow \underline{\underline{y'(x) = -\frac{x(9xy(x) + 10)}{3x^2 - 4y^3(x)}}} \quad \text{valid as long as the denominator is not 0}$$

For $x = 1$, $y(1) = 2$, one gets

$$y'(1) = -\frac{18 + 10}{3 - 32} = \frac{28}{29}$$

The slope of the tangent is $\frac{28}{29}$.

2. Clearly $F(0, \pm \sqrt[4]{5}) = -(\pm \sqrt[4]{5})^4 + 5 = 0$

^{2nd} $\Rightarrow (0, \pm \sqrt[4]{5})$ belong to the curve. 1

By evaluating \odot at 0, one further observe that the denominator is not 0

$(3 \cdot 0^2 - 4(\sqrt[4]{5})^3 \neq 0)$, and then $y'(0) = 0$

\Rightarrow the tangent at $(0, \pm \sqrt[4]{5})$ are the horizontal lines passing through $(0, \pm \sqrt[4]{5})$. 1

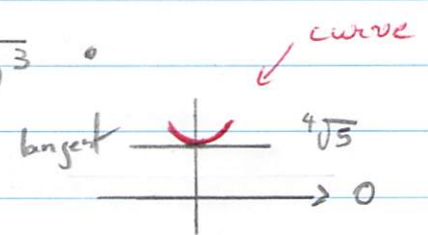
3. We need to compute y'' and evaluate it at

^{2nd} $x = 0$. However, since $y'(0) = 0$, all terms with $y'(0)$ can be directly disregarded.

$$y''(0) = \frac{-1(3 \cdot 0 \cdot y(0) + 10)(-4y^3(0)) - 0 - 0}{(-4y^3(0))^2}$$

$$= \frac{40}{16} \frac{y^3(0)}{y^6(0)} = \frac{5}{2} \frac{1}{y(0)^3}$$

At $(0, \sqrt[4]{5})$, $y''(0) > 0 \Rightarrow$



at $(0, -\sqrt[4]{5})$, $y''(0) < 0 \Rightarrow$

