Homework 12

Exercise 1 *Prove the following statement (ratio test) :*
Let $(a_j)_{j=1}^{\infty}$ be a sequence with positive terms only. Assume that there exists $c \in (0,1)$ and $N \in \mathbb{N}$ such that

$$\frac{a_{j+1}}{a_j} \leq c \qquad \forall j \geq N.$$

Then the corresponding series $a_1 + a_2 + a_3 + \dots$ is convergent.

Exercise 2 *1) Show that the series with generic term $a_j = \frac{j}{3^j}$ is convergent, in other terms show that*

$$\sum_{j=1}^{\infty} \frac{j}{3^j} < \infty.$$

2) Show that the series with generic term $a_j = (-1)^j \frac{j^3}{3^j}$ is absolutely convergent.

Exercise 3 *For $f : [1, \infty) \to \mathbb{R}_+$ decreasing, prove the following statement (integral test) :*
The series $f(1) + f(2) + f(3) + \dots$ is convergent if and only if $\lim_{M\to\infty} \int_1^M f(x)\,\mathrm{d}x$ is convergent, *or in a more simple form show that*

$$\sum_{j=1}^{\infty} f(j) < \infty \quad \Longleftrightarrow \quad \int_1^{\infty} f(x)\,\mathrm{d}x < \infty.$$

Exercise 4 *For any $\varepsilon > 0$ show that the series with generic term $a_j = j^{-1-\varepsilon}$ is convergent.*