Report

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December 2022

1 Proof of Lemma 4.2.5

Lemma 4.2.5: *If L is a semi-simple Lie algebra, its adjoint representation is faithful, namely $ad_X \neq ad_Y$ whenever $X \neq Y$. In addition, if L is simple, then its adjoint representation is irreducible.*

Let's look at the semi-simple situation first. Using Theorem 4.2.4, we can say this L's Killing form is nondegenerate, or $Det\big((g_{jk})\big) \neq 0$. According to the fourth property of Exercise 4.2.3, we can always choose a new basis such that the X, Y we are considering are two basis vectors. We now suppose there exist two basis vectors $X_1 \neq X_2$, such that $ad_{X_1} = ad_{X_2}$. For convenience, we arrange X_1 and X_2 to be the first two vectors in the basis vectors, and write down the explicit form of the matrix g_jk:

$$g_{jk} = \begin{bmatrix} Tr(ad_{X_1}ad_{X_1}) & Tr(ad_{X_1}ad_{X_1}) & Tr(ad_{X_1}ad_{X_3}) & ... & Tr(ad_{X_1}ad_{X_d}) \\ Tr(ad_{X_1}ad_{X_1}) & Tr(ad_{X_1}ad_{X_1}) & Tr(ad_{X_1}ad_{X_3}) & ... & Tr(ad_{X_1}ad_{X_d}) \\ . & & & & \\ . & & & & \\ . & & & & \end{bmatrix}, \tag{1}$$

where we have changed all the ad_{X_2} into ad_{X_1}. We can easily see that the first row and the second row are completely the same, thus we can conclude that the determinant of this matrix is 0, which is in contradiction with the condition that $Det\big((g_{jk})\big) \neq 0$. Thus, we must have $ad_X \neq ad_Y$ whenever $X \neq Y$.

The simple case is also simple. In fact, if the adjoint representation of L (L, ad) is reducible, then there exists at least one non-trivial proper subrepresentation (L', ad) such that $ad_X L' = [X, L'] = L'$ for any $X \in L$. This is in contradiction with the definition of simple.