Report

Li Yucheng

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1 Proof of Theorem 2.5.6 Selection Rule

Let $(\mathscr{H}, U)$ be a unitary representation which can be decomposed into $\mathscr{H} = \oplus_j \nu_j \mathscr{H}^j, U = \oplus_j \nu_j U^j$. Then, we define $\mathscr{U}$ by $\mathscr{U}(a)T := U(a)TU(a)^{-1}$. Note that $\mathscr{U} : G \to \mathscr{L}(\mathscr{B}(\mathscr{H}))$ defines a representation since $\mathscr{U}(e)T = U(e)TU(e)^{-1} = T \to \mathscr{U}(e) = \mathbb{1}$ and $\mathscr{U}(ab)T = U(ab)TU(ab)^{-1} = U(a)U(b)TU(b)^{-1}U(a)^{-1} = \mathscr{U}(a)U(b)TU(b)^{-1} = \mathscr{U}(a)\mathscr{U}(b)T \to \mathscr{U}(ab) = \mathscr{U}(a)\mathscr{U}(b)$.

Now we decompose this representation into irreducible representation $\mathscr{L}(\mathscr{B}(\mathscr{H})) = \oplus_l \mu_l \mathscr{L}^l$, $\mathscr{U} = \oplus_l \mu_l \mathscr{U}^l$, where $\mathscr{L}$ is the vector space of elements of $\mathscr{B}(\mathscr{H})$. Then, according to Lemma 2.3.6 (Schur lemma), we can define a similarity transformation $\tau_l : \mathscr{H}^l \to \mathscr{L}^l$ such that

$$\tau_l U^l(a) = \mathscr{U}(a)\tau_l. \tag{1}$$

Thus, for any $f \in \mathscr{H}^l$ and any $a \in G$, we have

$$\tau_l(U^l(a)f) = \mathscr{U}(a)\tau_l(f) = U(a)\tau_l(f)U(a)^{-1}. \tag{2}$$

Set $\mathscr{H}^{j,\nu}$ as one irreducible subspace of $\oplus_j \nu_j \mathscr{H}^j$ with $\nu \in 1, ..., \nu_j$. We define $\mathscr{M}_l^{j,\nu} := \{\tau_l(f)\psi | f \in \mathscr{H}^l, \psi \in \mathscr{H}^{j,\nu}\}$. This is invariant under the action of $U(a)$ for any $a \in G$ since $U(a)\mathscr{M}_l^{j,\nu} = U(a)\tau_l(\mathscr{H}^l)\mathscr{H}^{j,\nu} = \tau_l(U^l(a)\mathscr{H}^l)U(a)\mathscr{H}^{j,\nu} = \tau_l(\mathscr{H}^l)\mathscr{H}^{j,\nu} = \mathscr{M}_l^{j,\nu}$ due to eq. (2). This means that $\mathscr{M}_l^{j,\nu} = \oplus_j \nu_j' \mathscr{H}^j$, where $\nu_j' \leq \nu_j$.

Consider the tensor product $(\mathscr{H}^l \otimes \mathscr{H}^j, U^l \otimes U^j)$ being a representation of G. Define the map $Z : \mathscr{H}^l \otimes \mathscr{H}^j \to \mathscr{M}_l^{j,\nu}, \quad Z(f \otimes \psi) = \tau_l(f)\tilde{\psi}$, where $f \in \mathscr{H}^l, \psi \in \mathscr{H}^j$. The $\tilde{\psi}$ is ψ in $\mathscr{H}^{j,\nu}$. The image of Z is dense in $\mathscr{H}^{j,\nu}$, thus

$$\begin{aligned} Z(U^l(a) \otimes U^j(a) \quad f \otimes \psi) &= Z(U^l(a)f \otimes U^j(a)\psi) \\ &= \tau_l(U^l(a)f)\widetilde{U^j(a)}\psi \\ &= U(a)\tau_l(f)U(a)^{-1}U^j(a)\tilde{\psi} \\ &= U(a)\tau_l(f)\tilde{\psi} \\ &= U(a)Z(f \otimes \psi), \end{aligned}$$

where in the third equality we used eq. (2) and that U^j acts as $U(a)$ on $\mathscr{H}^{j,\nu}$. Note that the image of Z is on $\mathscr{M}_l^{j,\nu}$, we can then get the conclusion

$$Z \quad U^l \otimes U^j = U|_{\mathscr{M}_l^{j,\nu}} \quad Z. \tag{3}$$

Decompose $(\mathscr{H}^l \otimes \mathscr{H}^j, U^l \otimes U^j) = (\oplus_i \gamma_i \mathscr{H}^i, \oplus \gamma_i U^i)$. For one irreducible representation $(\mathscr{H}^i, U^i)$ of this decomposition, we define the restricted version of Z on $\mathscr{H}^i$ by $Z_i = Z|_{\mathscr{H}^i}$.

Thus, by replacing $U^l \otimes U^j$ with U^i and replacing Z with Z_i in eq. (3), we have $Z_i \quad U^i = U|_{\mathscr{M}_l^{j,\nu}} \quad Z_i$.

According to Proposition 2.15 in Amrein's note, if $ker Z_i \neq \mathscr{H}^i$, then Z_i is a similarity transformation, so that $(Z_i\mathscr{H}^i, U|_{Z_i\mathscr{H}^i})$ and $(\mathscr{H}^i, U^i)$ should be in the same class η_i. Thus, (recall that in paragraph 3 we decomposed $\mathscr{M}_l^{j,\nu}$ and there are ν_i representations of class η_i), there should be at least one subspace in $\mathscr{M}_l^{j,\nu}$ in the class η_i , which means that $\mathscr{H}^l \otimes \mathscr{H}^j$ and $\mathscr{M}_l^{j,\nu}$ have at least one representation in common (in the class η_i).

Then, when constructing the inner product $\langle \phi, \tau_l(f)\psi \rangle, \phi \in \mathscr{H}^i$ of an element of $\mathscr{H}^i$ with i being arbitrary and an element of $\mathscr{M}_l^{j,\nu}$, the result will depend on whether there is a representation of the same class in the decomposition of $(\mathscr{H}^l \otimes \mathscr{H}^j, U^l \otimes U^j)$. If there exists, then the inner product will not be 0 in general, but if there does not, it will be 0 due to the orthogonality of the decomposition.