Exercise 1.2.9

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1 Proof of Proposition 1.2.8

1.1

From the definition of $[a]_{G_0}$ above the Proposition 1.2.8, we know that $[a]_{G_0} = G_0 a$ and ${}_{G_0}[a] = aG_0$. Thus, if we have $[a]_{G_0} = {}_{G_0}[a]$, we get $aG_0 = G_0 a$. Multiplying both sides by a^{-1} from left, we have $aG_0 a^{-1} = G_0$ for an arbitrary $a \in G$, and this is just the definition of normal subgroup. If we suppose G_0 is a normal subgroup, we just need to reverse the procedure above and will find that $[a]_{G_0} = {}_{G_0}[a]$

1.2

First we can find that $[a]_{G_0}[b]_{G_0} = G_0 a G_0 b = G_0 G_0 ab = G_0 ab = [ab]_{G_0}$ by using the property of normal subgroup, which does define a product on the equivalence classes. Substitute $[b]_{G_0}$ by $[a]_{G_0}^{-1} := [a^{-1}]_{G_0}$, we get G_0. This is the unit element in $[a]_{G_0}$, which can be proved by $G_0[b]_{G_0} = G_0 G_0 b = G_0 b = [b]_{G_0}$, for any $b \in G$. Thus, these operations do define a group denoted by G/G_0.

2 Second part of exercise 1.2.9

We need to prove that

$$|G/G_0| = \frac{|G|}{|G_0|}. \tag{1}$$

Suppose there are r elements in G_0 and m classes in the quotient group G/G_0. From the definition of quotient group, if we represent each of the m classes by a representative v_i, we have

$$v_i \notin G_0 v_j, \quad i \neq j. \tag{2}$$

We can see that each class has the $G_0 v_i$ expression, so there are r elements in each class. Thus, the m classes has mr elements in total. This means that G has at least mr elements.

Now we suppose $|G| > mr$. Other than mr elements, there exists at least 1 element that does not belong to any of the m classes. Consider the class $[b]_{G_0} = {}_{G_0}[b] = bG_0 = G_0 b$, where b is an element other than the mr elements. This new class is not in the m classes, otherwise b itself would be in the m classes. But this indicates that together with this new class, we have $m+1$ classes in total, which contradicts to the assumption that we only have m classes.

Thus, we must conclude that $|G| = mr$, which proves the validity of (1).