On Schur's Lemma (Lemma 2.3.6)

This report aims to provide a proof for Schur's Lemma, which is an important result in representation theory.

The statement of Schur's Lemma is as follows:

Let (V, U) and (V', U') be irreducible representations of a group G. If there exists a linear map $\mathcal{T} : V \to V'$ such that for all $a \in G$,

$$\mathcal{T} U(a) = U'(a)\mathcal{T}, \tag{1}$$

then $\mathcal{T} = 0$ (the null map) or $\mathcal{T}$ is bijective (and as a consequence, $(V, U) \simeq (V', U')$).

1. Preliminary Proofs

Before we prove Schur's Lemma, let us prove that $\ker(\mathcal{T})$ and $\operatorname{im}(\mathcal{T})$ are subspaces of V and V', respectively.

Recall that subspaces are subsets of vector spaces that are closed under linear combinations.
In other words, if V_0 is a subspace of V, then for any $v_1, v_2 \in V_0$ and $\lambda_1, \lambda_2 \in \mathbb{C}$, $\lambda_1 v_1 + \lambda_2 v_2 \in V_0$.

First we prove $\ker(\mathcal{T})$ is a subspace of V. Let $k_1, k_2 \in \ker(\mathcal{T}) \subset V$, and set $k = \lambda_1 k_1 + \lambda_2 k_2$. Observe that:

$$\mathcal{T}(k) = \mathcal{T}(\lambda_1 k_1 + \lambda_2 k_2) = \mathcal{T}(\lambda_1 k_1) + \mathcal{T}(\lambda_2 k_2) = \lambda_1 \mathcal{T}(k_1) + \lambda_2 \mathcal{T}(k_2) = 0 \implies k \in \ker(\mathcal{T}).$$

Next, we prove that $\operatorname{im}(\mathcal{T})$ is a subspace of V'. Recall that $v' \in \operatorname{im}(\mathcal{T})$ means that $\exists v \in V$ such that $\mathcal{T}(v) = v'$.
Let $v'_1, v'_2 \in \operatorname{im}(\mathcal{T})$ such that $\exists v_1, v_2 \in V$ such that $\mathcal{T}(v_1) = v'_1$, $\mathcal{T}(v_2) = v'_2$. For $v' = \lambda_1 v'_1 + \lambda_2 v'_2$,

$$v' = \lambda_1 v'_1 + \lambda_2 v'_2 = \lambda_1 \mathcal{T}(v_1) + \lambda_2 \mathcal{T}(v_2) = \mathcal{T}(\lambda_1 v_1) + \mathcal{T}(\lambda_2 v_2) = \mathcal{T}(\lambda_1 v_1 + \lambda_2 v_2) \implies v' \in \operatorname{im}(\mathcal{T}).$$

As such, we have proven that $\ker(\mathcal{T})$ is a subspace of V, and $\operatorname{im}(\mathcal{T})$ a subspace of V'.

2. Proving Schur's Lemma

We divide the proof of Schur's Lemma into three parts:

2.1. Proving $\ker(\mathcal{T})$ is an invariant subspace of V

Let $k \in \ker(\mathcal{T})$, $a \in G$. Then observe that:

$$\mathcal{T} U(a)k = U'(a)\mathcal{T} k = U'(a)\, 0 = 0.$$

We obtain the first equality by (1) and the second one by definition of kernel. Notice that since $\mathcal{T}(U(a)k) = 0$, then $U(a)k \in \ker(\mathcal{T})$. As such, $U(a)\ker(\mathcal{T}) \subset \ker(\mathcal{T})$. Thus, we conclude that $\ker(\mathcal{T}) \subset V$ is an invariant subspace.

2.2. Proving $\operatorname{im}(\mathcal{T})$ is an invariant subspace of V'

Let $v' \in \operatorname{im}(\mathcal{T})$, $a \in G$. Let $v' = \mathcal{T} v$ for some $v \in V$. Observe that:

$$U'(a)v' = U'(a)\mathcal{T} v = \mathcal{T} U(a)v.$$

The first equality is due to $v' = \mathcal{T} v$, and the second one is by (1). Notice that $U(a) \in \mathcal{L}(V)$, so $U(a)v \in V$. Thus, $\mathcal{T} U(a)v \in \operatorname{im}(\mathcal{T})$. As such, $U'(a)\operatorname{im}(\mathcal{T}) \subset \operatorname{im}(\mathcal{T})$. Thus, we conclude that $\operatorname{im}(\mathcal{T}) \subset V'$ is an invariant subspace.

2.3. Drawing Conclusions on the nature of $\mathcal{T}$

(V, U) is an irreducible representation of G, so the only invariant subspaces of V is $\{0\}$ and V. From 2.1 we have that $\ker(\mathcal{T})$ is an invariant subspace of V, so $\ker(\mathcal{T}) = \{0\}$ or $\ker(\mathcal{T}) = V$. In the case that $\ker(\mathcal{T}) = V$, then $\mathcal{T} = 0$. In the case when $\ker(\mathcal{T}) = \{0\}$, we consider the image of $\mathcal{T}$.

Similarly, (V', U') is an irreducible representation of G, so $\operatorname{im}(\mathcal{T}) = V'$ or $\operatorname{im}(\mathcal{T}) = \{0\}$. If $\operatorname{im}(\mathcal{T}) = \{0\}$, then $\mathcal{T} = 0$ (the null map). If $\ker(\mathcal{T}) = \{0\}$ and $\operatorname{im}(\mathcal{T}) = V'$, then $\mathcal{T}$ is bijective. This concludes the proof of Schur's Lemma.