On Inner and Outer Semi-Direct Products (Exercises 1.3.5, 1.3.7)

This report aims to prove the equivalence of the inner and outer semi-direct products. Both semi-direct products might seem different on first glance. The concept of an inner semi-direct product is that if a group G has two subgroups N and H with $N \triangleleft G$ and meets a few conditions, then we say that the group G is an inner semi-direct product of the two subgroups. In other words, we use our knowledge of the group and subgroups to say something about the relationship between the group G and the two subgroups N and H.

On the other hand, the concept of the outer semi-direct product is to take two completely arbitrary groups and create a new group. While there might be countless ways to merge two sets together, the challenge here is to find a binary operation (the group operation) and an unary operation (the inverse) such that the merged set along with the two operations satisfy the three conditions of a group (and therefore forms a group).

1. Definition and uniqueness

A group G is called the *inner semi-direct product* of two of its subgroups N and H if they satisfy the following conditions:

1) N is a normal subgroup of G,

2) $N \cap H = \{e_G\}$ with e_G the identity element of G,

3) Each element g of G admits a decomposition $g = nh$ with $n \in N$ and $h \in H$.

If these three conditions are satisfied, we write $G = N \rtimes H$.

Proposition. Condition 2) above implies that the decomposition described in 3) is unique.

Proof. Let $g \in G$, $n_1, n_2 \in N$ and $h_1, h_2 \in G$. Suppose that $g = n_1 h_1 = n_2 h_2$. Then one has

$$n_2{}^{-1} n_1 h_1 = h_2 \Longleftrightarrow n_2^{-1} n_1 = h_2 h_1^{-1}.$$

The left-hand side of the equation is an element of N, while the right hand-side of the equation belongs to H. As per condition 2), the only element in common between N and H is e_G. Thus, $n_2^{-1} n_1 = h_2 h_1^{-1} = e_G$. Thus, $n_1 = n_2$ and $h_1 = h_2$, and we conclude that the decomposition of any element g is unique. □

On the other hand, the construction of the outer semi-direct product is more challenging. Let us have two arbitrary groups N and H. Consider a map $\psi : H \to \mathrm{Aut}(N)$, where $\mathrm{Aut}(N)$ is the group consisting of the set of all automorphisms of N along with the usual map composition as the group operation. We then define the *outer semi-direct product* of N and H as $N \rtimes_\psi H$ as the set $\{(n, h) \mid n \in N, h \in H\}$ with the (binary) operation

$$(n_1, h_1)(n_2, h_2) := (n_1[\psi(h_1)](n_2), h_1 h_2),$$

and the inverse $(n, h)^{-1} = ([\psi(h^{-1})](n^{-1}), h^{-1})$. We identify N with the set $\{(n, e_H) \mid n \in N\}$ and H with the set $\{(e_N, h) \mid h \in H\}$.

2. Proof that $N \rtimes_\psi H$ is a Group and $N \triangleleft (N \rtimes_\psi H)$

To prove $G = N \rtimes_\psi H$ is a group, we need to prove that G satisfies the three conditions of a group.

First, let us prove associativity. Let $(n_1, h_1), (n_2, h_2), (n_3, h_3) \in N \rtimes_\psi H$. Then,

$$\begin{aligned}
\big((n_1, h_1)(n_2, h_2)\big)(n_3, h_3) &= (n_1[\psi(h_1)](n_2), h_1 h_2)\,(n_3, h_3) \\
&= (n_1[\psi(h_1)](n_2)[\psi(h_1 h_2)](n_3), h_1 h_2 h_3) \\
(n_1, h_1)\big((n_2, h_2)(n_3, h_3)\big) &= (n_1, h_1)\,(n_2[\psi(h_2)](n_3), h_2 h_3) \\
&= (n_1[\psi(h_1)](n_2[\psi(h_2)](n_3)), h_1 h_2 h_3) \\
&= (n_1[\psi(h_1)](n_2)[\psi(h_1 h_2)](n_3), h_1 h_2 h_3)\,.
\end{aligned}$$

We have made use of the fact that $\psi(h)$ is an automorphism of N for all $h \in H$ and $[\psi(h_1)\psi(h_2)](n) = [\psi(h_1 h_2)](n)$. Next, we prove that $e_G = (e_N, e_H)$ is the identity element of G:

$$e_G(n,h) = (e_N, e_H)(n,h) = (e_N[\psi(e_H)](n), e_H h) = (e_N n, h) = (n,h),$$
$$(n,h)e_G = (n,h)(e_N, e_H) = (n[\psi(h)](e_N), h e_H) = (n e_N, h) = (n,h).$$

Here, we have used the fact that $[\psi(e_H)](n)$ is the identity map on N, and any automorphism must map e_N to e_N itself.

Then, we prove that the inverse defined on $N \rtimes_\psi H$ satisfies the inverse condition:

$$(n,h)(n,h)^{-1} = (n,h)([\psi(h^{-1})](n^{-1}), h^{-1}) = \left(n[\psi(h)\psi(h^{-1})](n^{-1}), hh^{-1}\right) = \left(n[\psi(e_H)](n^{-1}), e_H\right) = (e_N, e_H),$$
$$(n,h)^{-1}(n,h) = ([\psi(h^{-1})](n^{-1}), h^{-1})(n,h) = \left([\psi(h^{-1})](n^{-1})[\psi(h^{-1})](n), h^{-1}h\right) = \left([\psi(h^{-1})](e_N), e_H\right) = (e_N, e_H).$$

With these three conditions fulfilled, we have proven that $G = N \rtimes_\psi H$ is a group. Next, we prove that $N = \{n, e_H \mid n \in N\}$ is a normal subgroup of G. For all $(n,h) \in G$ and $(k, e_H) \in N$, one has:

$$\begin{aligned}(n,h)(k,e_H)(n,h)^{-1} &= (n,h)(k,e_H)([\psi(h^{-1})](n^{-1}), h^{-1}) = (n[\psi(h)](k), h e_H)\left([\psi(h^{-1})](n^{-1}), h^{-1}\right) \\ &= \left(n[\psi(h)](k)[\psi(h)\psi(h^{-1})](n^{-1}), hh^{-1}\right) = \left(n[\psi(h)](k)n^{-1}, e_H\right) \in N\,.\end{aligned}$$

As such, we conclude that $N \triangleleft (N \rtimes_\psi H)$.

3. The Equivalence of the Inner and Outer Semi-Direct Products

In this section, we will prove the equivalence of both concepts. To do this, we must prove this relationship both ways.

3.1. All Inner Semi-Direct Products are Outer Semi-Direct Products

Let us consider the automorphism of N defined by $[\psi(h)](n) = hnh^{-1}$. We now define the map

$$\begin{aligned}\psi : H &\longrightarrow \mathrm{Aut}(N) \\ h &\longmapsto [\psi(h)](n).\end{aligned}$$

Let us now prove that ψ is a homomorphism by proving $[\psi(h_1) \circ \psi(h_2)](n) = [\psi(h_1 h_2)](n)$:

$$[\psi(h_1) \circ \psi(h_2)](n) = [\psi(h_1)](h_2 n h_2^{-1}) = h_1(h_2 n h_2^{-1})h_1^{-1} = h_1 h_2 n h_2^{-1} h_1^{-1} = (h_1 h_2) n (h_1 h_2)^{-1} = [\psi(h_1 h_2)](n).$$

Let us now define an map $\phi : G \to N \rtimes_\psi H$ defined by $\phi(g) = \phi(nh) = (n,h)$.

Due to condition 3) on N and H (all $g \in G$ has unique decomposition $g = nh$ for $n \in N$ and $h \in H$), ϕ is well-defined for all g, and from that we also have that ϕ is surjective.

Since $e_{N \rtimes_\psi H} = (e_G, e_G)$, we have that $\mathrm{Ker}(\phi) = \{e_G\}$, so we have that ϕ is injective (and thus bijective).

Finally, we prove that ϕ is a homomorphism by showing $\phi(g_1)\phi(g_2) = \phi(g_1 g_2)$:

$$\phi(g_1)\phi(g_2) = (n_1, h_1)(n_2, h_2) = (n_1[\psi(h_1)](n_2), h_1 h_2) = (n_1 h_1 n_2 h_1^{-1}, h_1 h_2) = \phi(n_1 h_1 n_2 h_2) = \phi(g_1 g_2).$$

Therefore, we have proven that ϕ is an isomorphism, so we conclude that $G \simeq N \rtimes_\psi H$

3.2. All Outer Semi-Direct Products are Inner Semi-Direct Products

Let us set $G = N \rtimes_\psi H$. Let us first prove that $G_N = \{(n, e_H) \mid n \in N\}$ and $G_H = \{(e_N, h) \mid h \in H\}$ are subgroups.

Let us first prove it for G_N. We have that $e_N \in N$, so $(e_N, e_H) = e_G \in G_N$. Then, we have for $n_1, n_2, n_3 \in N$:

$$\begin{aligned}\left((n_1, e_H)(n_2, e_H)\right)(n_3, e_H) &= (n_1[\psi(e_H)](n_2), e_H e_H)(n_3, e_H) = (n_1 n_2, e_H)(n_3, e_H) \\ &= (n_1 n_2[\psi(e_H)](n_3), e_H e_H) = (n_1 n_2 n_3, e_H) \\ &= (n_1, h_1)(n_2 n_3, e_H) = (n_1, h_1)\left((n_2, e_H)(n_3, e_H)\right)\end{aligned}$$

As such, we have proven associativity. Next, we prove the inverse condition:

$$(n, e_H)(n^{-1}, e_H) = (n[\psi(e_H)](n^{-1}), e_H e_H) = (nn^{-1}, e_H) = (e_N, e_H) = e_G.$$
$$(n^{-1}, e_H)(n, e_H) = (n^{-1}[\psi(e_H)](n), e_H e_H) = (n^{-1}n, e_H) = (e_N, e_H) = e_G.$$

Therefore, we conclude that G_N is a subgroup of G.

For G_H, observe that since $e_H \in H$, $(e_N, e_H) = e_G \in G_H$. Then, for $h_1, h_2, h_3 \in H$,

$$\begin{aligned}\Big((e_N, h_1)(e_N, h_2)\Big)(e_N, h_3) &= (e_N[\psi(h_1)](e_N), h_1h_2)(e_N, h_3) = (e_N, h_1h_2h_3)\\ &= (e_N[\psi(h_1)](e_N), h_1h_2h_3) = (e_N, h_1)(e_N, h_2h_3)\\ &= (e_N, h_1)(e_N[\psi(h_2)](e_N), h_2h_3) = (e_N, h_1)\Big((e_N, h_2)(e_N, h_3)\Big)\end{aligned}$$

As such, we have proven associativity. Next, we prove the inverse condition:

$$(e_N, h)(e_N, h^{-1}) = (e_N[\psi(h)](e_N), hh^{-1}) = (e_N, e_H) = e_G,$$
$$(e_N, h^{-1})(e_N, h) = (e_N[\psi(h^{-1})](e_N), h^{-1}h) = (e_N, e_H) = e_G.$$

As such, we conclude that G_H is also a subgroup of G.

All we have to do is to prove that G_N and G_H satisfy the three conditions on inner semi-direct products stipulated above.

First, we prove that $G_N \triangleleft G$. Let $(k, e_H) \in G_N$ and $(n, h) \in G$. Then,

$$\begin{aligned}(n, h)(k, e_H)(n, h)^{-1} &= (n[\psi(h)](k), h)\left([\psi(h^{-1}](n^{-1}), h^{-1}\right) = \left(n[\psi(h)](k) \cdot [\psi(h) \circ \psi(h^{-1})](n^{-1}), hh^{-1}\right)\\ &= \left(n[\psi(h)](k) \cdot [\psi(e_H)](n^{-1}), e_H\right) = \left(n[\psi(h)](k)n^{-1}, e_H\right) \in G_N.\end{aligned}$$

Therefore, G_N is a normal subgroup of G.

Next, we observe that $G_N \cap G_H = \{e_G\}$. Finally, we observe that $\forall (n, h) \in G$:

$$(n, e_H)(e_N, h) = (n[\psi(e_H)](e_N), e_H h) = (ne_N, e_H h) = (n, h) = g.$$

Observe that $(n, e_H) \in G_N$ and $(e_N, h) \in G_H$. Thus, the three conditions are fulfilled. As such, $G = G_N \rtimes G_H$.

Therefore, we conclude that the inner semi-direct product is equivalent to the outer semi-direct product.