About the induced representation

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Exercise 5.4.1.

Let G be a finite group, and let G_0 be a subgroup of G. We also assume that $(\mathcal{V}, U)$ is a finite dimensional representation of G_0, with $\dim(\mathcal{V}) = n$. Check that the pair $(\mathcal{W}, \mathcal{U})$ given by

$$\mathcal{W} := \left\{ f : G \to \mathcal{V} \middle| f(aa_0) = U(a_0^{-1})f(a), \forall a \in G, a_0 \in G_0 \right\},$$

$$[\mathcal{U}(b)f](a) := f(b^{-1}a), \text{ for every } a, b \in G,$$

define a representation of G in $\mathcal{W}$.

1. $\mathcal{W}$ is a vector space

For $f, g \in \mathcal{W}$, scalar λ (in either $\mathbb{R}$ or $\mathbb{C}$), $a \in G$, and $a_0 \in G_0$, we have

$$(f+g)(aa_0) = f(aa_0) + g(aa_0) = U(a_0^{-1})f(a) + U(a_0^{-1})g(a) = U(a_0^{-1})(f(a) + g(a)) = U(a_0^{-1})(f+g)(a),$$

$$(\lambda f)(aa_0) = \lambda f(aa_0) = \lambda \left(U(a_0^{-1})f(a)\right) = U(a_0^{-1})\left(\lambda f(a)\right) = U(a_0^{-1})(\lambda f)(a).$$

Hence, for every $f, g \in \mathcal{W}$ and every scalar λ

$$f + g \in \mathcal{W},$$

$$\lambda f \in \mathcal{W}.$$

Thus, $\mathcal{W}$ is a vector space.

2. $\mathcal{U}$ is a map from G to $\mathcal{L}(\mathcal{W})$

For $f \in \mathcal{W}$, $a, b \in G$, and $a_0 \in G_0$, we have

$$[\mathcal{U}(b)f](aa_0) = f(b^{-1}aa_0) = U(a_0^{-1})f(b^{-1}a) = U(a_0^{-1})[\mathcal{U}(b)f](a).$$

Hence, $\mathcal{U}(b)f \in \mathcal{W}$ for every $f \in \mathcal{W}$, which implies that $\mathcal{U}(b)$ is a map from $\mathcal{W}$ to $\mathcal{W}$ for any $b \in G$.

Also, for $f, g \in \mathcal{W}$, scalar λ, and $a, b \in G$, we have

$$[\mathcal{U}(b)(f+g)](a) = (f+g)(b^{-1}a) = f(b^{-1}a) + g(b^{-1}a) = [\mathcal{U}(b)f](a) + [\mathcal{U}(b)g](a) = [\mathcal{U}(b)f + \mathcal{U}(b)g](a),$$

$$[\mathcal{U}(b)(\lambda f)](a) = (\lambda f)(b^{-1}a) = \lambda f(b^{-1}a) = \lambda [\mathcal{U}(b)f](a) = [\lambda \mathcal{U}(b)f](a).$$

Hence, for every $f, g \in \mathcal{W}$ and every scalar λ,

$$\mathcal{U}(b)(f+g) = \mathcal{U}(b)f + \mathcal{U}(b)g,$$
$$\mathcal{U}(b)(\lambda f) = \lambda \mathcal{U}(b)f.$$

Thus, $\mathcal{U}(b)$ is a linear map on $\mathcal{W}$ for any $b \in G$. Consequently, $\mathcal{U}$ is a map from G to $\mathcal{L}(\mathcal{W})$.

3. $(\mathcal{W}, \mathcal{U})$ is a representation of G

For $f \in \mathcal{W}$ and $a, b, c \in G$, we have

$$[\mathcal{U}(bc)f](a) = f\left((bc)^{-1}a\right) = f\left(\left(c^{-1}b^{-1}\right)a\right) = f\left(c^{-1}\left(b^{-1}a\right)\right) = [\mathcal{U}(c)f]\left(b^{-1}a\right) = [\mathcal{U}(b)\mathcal{U}(c)f](a).$$

Hence, $\mathcal{U}(bc)f = \mathcal{U}(b)\mathcal{U}(c)f$ for any $f \in \mathcal{W}$, and therefore, $\mathcal{U}(bc) = \mathcal{U}(b)\mathcal{U}(c)$ for any $b, c \in G$. Also, for the identity element $e \in G$, we have

$$[\mathcal{U}(e)f](a) = f(e^{-1}a) = f(ea) = f(a), \quad \text{for every } a \in G.$$

Thus, $\mathcal{U}(e)f = f$ for any $f \in \mathcal{W}$, or $\mathcal{U}(e) = \mathbb{1}$. As a result, $(\mathcal{W}, \mathcal{U})$ is a representation of G.