On characters of unitary and irreducible representations of a finite group

SML - Groups and Their Representations

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Exercise 2.4.5

Let $(\mathcal{H}^k, U^k)$ and $(\mathcal{H}^\ell, U^\ell)$ be two unitary and irreducible representations of a finite group G, with respective characters denoted by χ^k and χ^ℓ. Then,

$$\frac{1}{|G|}\sum_{a\in G}\overline{\chi^k(a)}\chi^\ell(a) = \begin{cases} 1, & \textit{if } (\mathcal{H}^k, U^k) \simeq (\mathcal{H}^\ell, U^\ell) \\ 0, & \textit{otherwise} \end{cases} \quad . \tag{1}$$

Proof. When $(\mathcal{H}^k, U^k)$ and $(\mathcal{H}^\ell, U^\ell)$ are identical or inequivalent, we have this equality

$$\frac{1}{|G|}\sum_{a\in G} U^\ell_{rs}(a)\overline{U^k_{ij}(a)} = \frac{1}{n_k}\delta_{k\ell}\delta_{sj}\delta_{ri}, \qquad \text{for } i,j\in\{1,\dots,n_k\} \text{ and } r,s\in\{1,\dots,n_\ell\}. \tag{2}$$

Then, we have

$$\begin{aligned}\frac{1}{|G|}\sum_{a\in G}\overline{\chi^k(a)}\chi^\ell(a) &= \frac{1}{|G|}\sum_{a\in G}\overline{\mathrm{Tr}\left(U^k(a)\right)}\mathrm{Tr}\left(U^\ell(a)\right) \\ &= \frac{1}{|G|}\sum_{a\in G}\overline{\sum_{i=1}^{n_k}U^k_{ii}(a)}\sum_{j=1}^{n_\ell}U^\ell_{jj}(a) \\ &= \sum_{i=1}^{n_k}\sum_{j=1}^{n_\ell}\frac{1}{|G|}\sum_{a\in G}\overline{U^k_{ii}(a)}U^\ell_{jj}(a) \\ &= \sum_{i=1}^{n_k}\sum_{j=1}^{n_\ell}\frac{1}{n_k}\delta_{k\ell}\delta_{ji}\delta_{ji} \\ &= \frac{1}{n_k}\delta_{k\ell}\min\{n_k, n_\ell\}\end{aligned}$$

For the factor $1/n_k$, we can choose either n_k or n_ℓ because $\delta_{k\ell}$ on the right hand side of (2) is nonzero when $k=\ell$. Hence, we can choose it so that it cancels $\min\{n_k, n_\ell\}$. Thus,

$$\frac{1}{|G|}\sum_{a\in G}\overline{\chi^k(a)}\chi^\ell(a) = \delta_{k\ell}.$$

In addition, consider the case that $(\mathcal{H}^k, U^k)$ and $(\mathcal{H}^\ell, U^\ell)$ are not identical but equivalent. In that case, $\chi^k = \chi^\ell$, which means that the sum on the right hand side of (1) when $(\mathcal{H}^k, U^k) \simeq (\mathcal{H}^\ell, U^\ell)$ and $(\mathcal{H}^k, U^k) \neq (\mathcal{H}^\ell, U^\ell)$ is the same as when $(\mathcal{H}^k, U^k) = (\mathcal{H}^\ell, U^\ell)$. Hence,

$$\frac{1}{|G|}\sum_{a\in G}\overline{\chi^k(a)}\chi^\ell(a) = \begin{cases} 1, & \text{if } (\mathcal{H}^k, U^k) \simeq (\mathcal{H}^\ell, U^\ell) \\ 0, & \text{otherwise} \end{cases} \qquad \text{Q.E.D.}$$