A Proof of Maschke's theorem on vector space

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Theorem. *If G is a finite group, any finite dimensional representation $(\mathcal{H}, U)$ in a vector space is completely reducible.*

Remark 1. $(\mathcal{H}, U)$ in a finite dimensional vector space is completely reducible if $\mathcal{H}$ can write by using finite number of irreducible representation $\mathcal{H}_k$ as follows:

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_n.$$

Knowing that if the following proposition is true, then the theorem above is also true.

Proposition. *Let $(\mathcal{H}, U)$ be reducible and finite dimensional. For all U-invariant subspace $\mathcal{H}_1 \subseteq \mathcal{H}$, ($\mathcal{H}_1$ is U-invariant subspace $\Leftrightarrow \boldsymbol{h} \in \mathcal{H}_1 \Rightarrow U\boldsymbol{h} \in \mathcal{H}_1$)*
There exists U-invariant subspace $\mathcal{H}_2$ satisfying the following condition:

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Firstly, let us prove the theorem assuming the proposition is true by using induction.
When $\dim \mathcal{H} = 1$, it is true since $\mathcal{H}$ is irreducible. Assuming that $\dim \mathcal{H} \geq 2$ and the proposition is true until the dimension of $\mathcal{H} = n-1$. Then let us prove the proposition is also true when the dimension of $\mathcal{H} = n$. There exists non-trivial U-invariant subspace $\mathcal{H}_1$, since $(\mathcal{H}, U)$ is reducible. And we can take U-invariant subspace $\mathcal{H}_2$ satisfying $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$ by the proposition. Because $\dim \mathcal{H}_1 < \dim \mathcal{H}$, $\dim \mathcal{H}_2 < \dim \mathcal{H}$, from the assumption of the induction, both $\mathcal{H}_1$ and $\mathcal{H}_2$ are the direct sum of irreducible representations of finite numbers We can see the fact by using the proposition repeatedly until each elements of direct sum become irreducible representation. Thus, $\mathcal{H}$ is also a completely reducible representation. ■

Then, let us prove the proposition.

Proof. Let us take U-invariant subspace $\mathcal{H}_1$, and consider $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$.
To give the above direct sum decomposition as a linear space, all we need to do is construct $P \in \mathrm{End}(\mathcal{H})$ (i.e. $\mathrm{End}(\mathcal{H}) \coloneqq \{f \mid f : \mathcal{H} \to \mathcal{H}\}$) with $P^2 = P$, $\mathrm{Im}(P) = \mathcal{H}_1$. Because by defining $\mathcal{H}_2 \coloneqq \mathrm{Ker}(P)$ and $P' \coloneqq \mathrm{Id}_V - P$, we can make orthogonal projection system [*1] P, P', where $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ holds. In

[*1] There exists the following theorem.
When linear maps $P_1, \cdots, P_k$ satisfy the following conditions
(i)$P_i^2 = P_i (1 \leq i \leq k)$
(ii)$i \neq k \Rightarrow P_i P_j = O$

addition, for $\mathcal{H}_2$ to be G-invariant, P should be G-homomorphism. [*2]

Let P_0 be the projection to $\mathcal{H}_1$ of the above direct sum decomposition. We also modified the P_0, we defined P as follows

$$P := \frac{1}{|G|}\sum_{a\in G} U(a)^{-1}P_0\,U(a) \in \mathrm{End}(\mathcal{H}_1)$$

We can see that $\mathrm{Im}(P) \subseteq \mathcal{H}_1$ since $\mathcal{H}_1$ is G-invariant.
For $\boldsymbol{h}_1 \in \mathcal{H}_1$,

$$\begin{aligned}
P\boldsymbol{h}_1 &= \frac{1}{|G|}\sum_{a\in G} U(a)^{-1}P_0\,U(a)\boldsymbol{h}_1 \\
&= \frac{1}{|G|}\sum_{a\in G} U(a)^{-1}U(a)\boldsymbol{h}_1 \quad (\because U(a)\boldsymbol{h}_1 \in \mathcal{H}_1) \\
&= \frac{1}{|G|}\sum_{a\in G} U(a^{-1}a)\boldsymbol{h}_1 \\
&= \frac{1}{|G|}\sum_{a\in G} \boldsymbol{h}_1 \\
&= \boldsymbol{h}_1
\end{aligned}$$

Thus, $\mathcal{H}_1 \subseteq \mathrm{Im}(P) = \mathcal{H}_1 \therefore \mathrm{Im}(P) = \mathcal{H}_1$.
Also, because of $P\boldsymbol{h}_1 = \boldsymbol{h}_1$ for all $\boldsymbol{h}_1 \in \mathcal{H}_1$, $P(P\boldsymbol{h}) = P(P(\boldsymbol{h}_1 + \boldsymbol{h}_2)) = P\boldsymbol{h}_1$ where $\boldsymbol{h}_2 \in \mathrm{Ker}(P)$, Thus $P^2 = P$.
Finally, let us prove that P is G-homomorphism. For $b \in G$

$$\begin{aligned}
PU(b) &= \frac{1}{|G|}\sum_{a\in G} U(a^{-1})P_0U(a)U(b) \\
&= \frac{1}{|G|}\sum_{a\in G} U(a^{-1})P_0U(ab) \\
&= \frac{1}{|G|}\sum_{a\in G} U(b)U(b^{-1})U(a^{-1})P_0U(ab) \quad (\text{Using the fact that}\, U(b)U(b^{-1}) = \mathrm{Id}_V) \\
&= \frac{1}{|G|}\sum_{a\in G} U(b)U((ab)^{-1})P_0U(ab) \\
&= U(b)\frac{1}{|G|}\sum_{a\in G} U(a^{-1})P_0U(a) \quad \cdots(*) \\
&= U(b)P
\end{aligned}$$

(iii)$P_1 + \cdots P_k = \mathrm{Id}_V$
By setting $\mathcal{H}_i := \mathrm{Im}(P_i)$, the following direct sum decomposition holds

$$\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_k$$

[*2] Since

$$\begin{aligned}
\forall\boldsymbol{h}_2 \in \mathcal{H}_2;\, U(a)\boldsymbol{h}_2 \in \mathcal{H}_2 &\Leftrightarrow \forall\boldsymbol{h}_2 \in \mathrm{Ker}(P);\, U(a)\boldsymbol{h}_2 \in \mathrm{Ker}(P) \\
&\Leftrightarrow P\boldsymbol{h}_2 = 0 \Rightarrow P(U(a)\boldsymbol{h}_2) = 0
\end{aligned}$$

$(*)$ I use the following fact:
Let V be linear space and F be a function from group G to V.
All $b \in G$, satisfy the following equation:

$$\sum_{a \in G} F(a) = \sum_{a \in G} F(ba) = \sum_{a \in G} F(ab) = \sum_{a \in G} F(a^{-1})$$

Because of the above, $\mathcal{H}_2$ satisfies $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and it is G-invariant. □

References

[1] Serge Richard. *Groups and their representations.* 2022

[2] Gaku Ikeda. Tensor *Algebra and Representation Theory.* 2020