Non-Hausdorff Topological Spaces and Hausdorff implies T_1

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Example 1

Let $X = \{a, b, c, d\}$ to be a set with the topology $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$. Let us check the properties of a Hausdorff space. The set of all open neighborhoods of element $a \in X$ is given by $\nu_a = \{\{a\}, \{a, b\}, \{a, b, c\}, X\}$. Meanwhile, for the element $b \in X$, the set of all open neighborhoods of b is given by $\nu_b = \{\{a, b\}, \{a, b, c\}, X\}$. As we can observe, the intersection of any of the defined open neighborhoods is nonempty. Hence, there does not exist any open neighborhoods $V_1 \in \nu_a$ and $V_2 \in \nu_b$ such that $V_1 \cap V_2 = \emptyset$. Therefore, the topological space (X, τ) is not a Hausdorff space.

Example 2

Let us consider an interesting case in which for a topological space (X, τ), the singleton set $\{x\}$ is closed for every $x \in X$. Let p be such that $p \notin [0, 1]$ and let $X = [0, 1] \cup \{p\}$ be topologized in such a way that $[0, 1]$ has the subspace topology from $\mathbb{R}$ and the neighborhoods of p have the form of $]1 - \varepsilon, 1[\cup p$ for $0 < \varepsilon < 1$. Before proving the topological space (X, τ) is not a Hausdorff space, let us prove that the singleton sets $\{x\}$ are closed. Observe that the interval $(0, 1]$ is an open set on the defined topological space (X, τ) as we have set $p \notin [0, 1]$. Then, one also has $[0, 1)$ an open set on the topology. Thus, the singleton $\{0\}$ is closed on $[0, 1]$ as $\{0\} = (0, 1]^{\mathsf{c}}$. With the same argument, the singleton $\{1\}$ is closed on $[0, 1]$ as $\{0\} = [0, 1)^{\mathsf{c}}$. Observe that the union of two open sets $[0, x) \cup (x, 1]$ is also open on $[0, 1]$ for some $x \in (0, 1)$. Hence, the singleton $\{x\}$ is closed on $[0, 1]$ as $\{x\} = ([0, x) \cup (x, 1])^{\mathsf{c}}, \forall x \in (0, 1)$. Finally, we know that the singleton set $\{p\}$ is closed on X as $[0, 1]$ is an open set in our topology and we have $\{p\} = [0, 1]^{\mathsf{c}}$. Since $\{p\}$ is closed on X and $p \notin [0, 1]$, we have $\{0\}$, $\{1\}$, and singleton sets $\{x\}$ closed on X as well for all $x \in (0, 1)$. Therefore, the singleton set $\{x\}$ is closed for every $x \in X$. A topological space that has all the singleton sets closed is called **$\mathbf{T_1}$ space or Fréchet space.** More precisely, T_1 space is defined as follows

Definition 1

A topological space is called T_1 space if it satisfies the following equivalent conditions ([1]):

1. Given two distinct points $x, y \in X$, there exists an open subset U of X such that $x \in U$ and $y \notin U$.

2. For every $x \in X$, the singleton set $\{x\}$ is a closed subset.

3. For every $x \in X$, the intersection of all open subsets of X containing $\{x\}$ is precisely $\{x\}$.

Therefore, **Hausdorff spaces implies $\mathbf{T_1}$** which follows from the definition of Hausdorff spaces. Now let us consider the Hausdorff space property of the topological space. One can argue that (X, τ) topological space is not a Hausdorff space as there does not exist a disjoint open neighborhood of 1 and p, namely there does not exist open neighborhoods $1 \in V_1$ and $p \in V_2$ such that $V_1 \cap V_2 = \emptyset$.

Remark

The sets inside the topological space defined in **Example 1** are all closed since a finite subset in some metric space is closed. Consider a finite set $A = \{x_1, x_2, \ldots, x_n\}$ since any single point in A is closed as we can construct the same proof construction in **Example 2** and one has $A = \cup_{j=1}^{n} \{x_j\}$. Since A itself is a finite union of closed sets, A is therefore closed.

References

[1] James R Munkres. *Topology*, volume 2. Prentice hall Upper Saddle River, 2000.