Representation of a Finite Group is Equivalent to the Unitary Representation

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Special Mathematics Lecture: Groups and their representations (Fall 2022)

Proposition 2.2.6

Let G be a finite group, and let $(\mathcal{H}, U)$ be a representation of G in a Hilbert Space $\mathcal{H}$. Then, U is equivalent to unitary representation $(\mathcal{H}', U')$.

<u>Proof:</u> Let $\langle\, , \rangle$ be the scalar product on Hilbert space $\mathcal{H}$ and $(\mathcal{H}, U')$ be a representation of G such that U' is isomorphic to map $U : G \to \mathcal{L}(\mathcal{H})$. Let us define the following expression

$$\langle f', f\rangle' = \frac{1}{|G|}\sum_{a\in G}\langle U'(a)f', U'(a)f\rangle$$

for $f, f' \in \mathcal{H}$. *This trick is also widely known as Weyl's Unitary Trick.* Note that, the definition above can only be satisfied by the finite groups as they only have finite numbers of elements. One can argue that $\langle\, , \rangle'$ is linear in the second argument by virtue of $\langle\, , \rangle$ is also linear in the second argument. For $f, g, h \in \mathcal{H}$, the linearity is given by

$$\begin{aligned}\langle f, \lambda g + h\rangle' &= \frac{1}{|G|}\sum_{a\in G}\langle U'(a)f, U'(a)(\lambda g + h)\rangle \\ &= \frac{1}{|G|}\sum_{a\in G}\lambda\langle U'(a)f, U'(a)g\rangle + \frac{1}{|G|}\sum_{a\in G}\langle U'(a)f, U'(a)h\rangle \\ &= \lambda\langle f, g\rangle' + \langle f, h\rangle'.\end{aligned}$$

Similarly, one can show that $\langle\, , \rangle'$ is anti-linear in the first argument. Additionally, $\langle\, , \rangle'$ preserves the positive-definite property as the sum of positive terms always gives a positive solution unless all of the terms are equal to zero. If all of the summands are equal to zero, then that means for $f \in \mathcal{H}$, $\langle f, f\rangle' = 0 \iff f = 0$ as a result of the equality of scalar product $\langle\, , \rangle$. Therefore, $\langle\, , \rangle'$ can also be defined as a scalar product. We define $\mathcal{H}'$ as the vector space $\mathcal{H}$ endowed with the scalar product $\langle\, , \rangle'$. Let us show that U' is unitary with respect to $\langle\, , \rangle'$

$$\langle U'(a)f, U'(a)f'\rangle' = \frac{1}{|G|}\sum_{b\in G}\langle U'(ba)f, U'(ba)f'\rangle = \frac{1}{|G|}\sum_{c\in G}\langle U'(c)f, U'(c)f'\rangle = \langle f, f'\rangle'.$$

Hence, U' is unitary with respect to $\langle\, , \rangle'$. Let $(\mathcal{H}, U)$ and $(\mathcal{H}, U')$ be representations of group G.

Since U' is isomorphic to U, then there exists a bijective linear map $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ such that the linear map $\mathcal{T}$ acts as the change of basis between two linear representations $(\mathcal{H}, U)$ and $(\mathcal{H}, U')$. Hence, one has

$$U(a) = \mathcal{T} U'(a) \mathcal{T}^{-1}.$$

By equipping $\mathcal{H}$ with the new scalar product $\langle \, , \rangle'$, we get that U is equivalent to the unitary representation $(\mathcal{H}', U')$.