$\mathcal{P}$ is Isomorphic to the Semi-direct Product $T(4) \rtimes \mathcal{L}$

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Special Mathematics Lecture: Groups and their representations (Fall 2022)

Exercise 1.5.6

Check that $(T(4), \mathbb{1})$ *is a normal subgroup of* $\mathcal{P}$*, and that* $\mathcal{P}$ *is isomorphic to the semi-direct product* $T(4) \rtimes \mathcal{L}$.

<u>**Proof:**</u> Let us first check that $(T(4), \mathbb{1})$ is a normal subgroup of $\mathcal{P}$. Consider the element $(b, \Lambda) \in \mathcal{P}$. Then, for all element $(b, \Lambda) \in \mathcal{P}$, one has

$$\begin{aligned}
(b,\Lambda)(T(4),\mathbb{1})(b,\Lambda)^{-1} &= (b,\Lambda)(T(4),\mathbb{1})(-\Lambda^{-1}b,\Lambda^{-1}) \\
&= (b+\Lambda T(4),\Lambda)(-\Lambda^{-1}b,\Lambda^{-1}) \\
&= (b+\Lambda T(4)+\Lambda(-\Lambda^{-1}b),\Lambda\Lambda^{-1}) \\
&= (\Lambda T(4),\mathbb{1}) \\
&= (T(4),\mathbb{1}).
\end{aligned}$$

We argued that $\Lambda T(4) = T(4)$ with $\Lambda \in \mathcal{L}$ as the matrix Λ in the Lorentz group preserves the bilinear map defined in **Definition 1.5.3** and the map $x \mapsto \Lambda x$ with $x \in \mathbb{R}^4$ is bijective as we can show by considering its determinant. From the matrix form of the Lorentz group's relation, one has

$$\det(\Lambda^T g\Lambda) = \det(\Lambda^T)\det(g)\det(\Lambda) = \det(g) \implies \det(\Lambda^T)\det(\Lambda) = 1.$$

Therefore, $(T(4), \mathbb{1})$ is a normal subgroup of $\mathcal{P}$. From **Definition 1.5.5**, the elements of $\mathcal{P}$ is given by (b, Λ) with $b \in T(4)$ and $\Lambda \in \mathcal{L}$. One can identify $T(4) \equiv \{(b, \mathbb{1}) \mid b \in T(4)\}$ and $\mathcal{L} \equiv \{(0, \Lambda) \mid \Lambda \in \mathcal{L}\}$. Let us show that the groups $T(4)$ and $\mathcal{L}$ indeed correspond to the sets. For $b, b' \in T(4)$, $\Lambda, \Lambda' \in \mathcal{L}$, and $(b, \mathbb{1}), (b', \mathbb{1}), (0, \Lambda), (0, \Lambda') \in \mathcal{P}$, one has

$$\begin{aligned}
(b,\mathbb{1})(b',\mathbb{1}) &= (b+\mathbb{1}b',\mathbb{1}) \\
&= (b+b',\mathbb{1}) \\
(0,\Lambda)(0,\Lambda') &= (0+\Lambda(0),\Lambda\Lambda') \\
&= (0,\Lambda\Lambda')
\end{aligned}$$

As we can see from the relations above, the multiplication in the Poincaré group corresponds to the addition in $T(4)$ and the product in $\mathcal{L}$. Therefore, the groups $T(4)$ and $\mathcal{L}$ indeed correspond to the sets. Observe that the intersection of both groups is given by

$$T(4) \cap \mathcal{L} = \{(b, \mathbb{1})\} \cap \{(0, \Lambda)\} = (0, \mathbb{1}) = e.$$

As we can see, the intersection of $T(4)$ and $\mathcal{L}$ is given by the identity element of $\mathcal{P}$. Lastly, we can observe that each element of $\mathcal{P}$ admits a decomposition consisting of elements from $T(4)$ and $\mathcal{L}$ as shown below

$$(b, \mathbb{1})(0, \Lambda) = (b + \mathbb{1}(0), \mathbb{1}\Lambda) = (b, \Lambda).$$

Hence, each element $(b, \Lambda) \in \mathcal{P}$ admits a decomposition $(b, \Lambda) = (b, \mathbb{1})(0, \Lambda)$ with $(b, \mathbb{1}) \in T(4)$ and $(0, \Lambda) \in \mathcal{L}$. As the three properties of the Inner semi-direct product have been fulfilled, then one can define the semi-direct product $T(4) \rtimes \mathcal{L}$ such that the Poincaré group $\mathcal{P}$ is isomorphic to the semi-direct product $T(4) \rtimes \mathcal{L}$.