Nguyen Quan Minh

Group Theory - SML

Question: prove $U(a)$, a representation (V, U) of a group G, is injective if and only if $U(a)$ is faithful

Answer:

- First, we prove $U(a)$ is faithful $\rightarrow$ $U(a)$ is injective

  We know that $U(a)$ is faithful $\Leftrightarrow U(a) \neq 1 \; \forall a \in G \setminus \{e\}$ and we have $U(e) = 1$

  To prove $U(a)$ is injective, we tried to prove that if $U(a) = U(b) \Rightarrow a = b$

  we see that $a, b \in G$ so $a.a^{-1} = e$.

  If $U(a) = U(b)$

  $\Leftrightarrow U(a).U(a^{-1}) = U(b).U(a^{-1})$

  $\Leftrightarrow U(a.a^{-1}) = U(ba^{-1})$

  $\Leftrightarrow U(e) = U(b.a^{-1}) = 1$

  Because $U(a) \neq 1 \; \forall a \in G \setminus \{e\} \rightarrow e = b.a^{-1} \Rightarrow a = b \rightarrow U(a)$ is injective

  $\rightarrow$ Q.E.D

- Now, we prove $U(a)$ is injective $\rightarrow$ $U(a)$ is faithful

  If $U(a)$ is injective $\rightarrow U(a) = U(b) \rightarrow a = b$

  If $U(a) = U(e) = 1 \Leftrightarrow a = e$

  $\rightarrow U(a) \neq 1 \; \forall a \in G \setminus \{e\} \rightarrow U(a)$ is faithful

  $\rightarrow$ Q.E.D

So the representation (V, U) of group G is faithful if and only if $U(a)$ is injective.