Definition of Lie Group and proof that the identity component is a normal subgroup

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1 Introduction

In this report I will first explain the definition of a Lie Group (Exercise 3.2.2) and then show that the identity component G_0 of any Lie Group is a normal subgroup (Exercise 3.3.2).

2 Exercise 3.2.2: Definition of a Lie Group

The definition of a Lie Group as given in [1] is:

Definition of Lie Group

A lie group G is a group that is also a finite dimensional smooth manifold, for which the group law and the inversion are smooth maps.

The meaning of the group law and the inversion being smooth maps only makes sense in a metric space. Luckily, since G is also a finite dimensional smooth manifold, there exist local charts around every element a, through which we have a notion of distance. Let us consider the local charts around a, b and the product ab. This means the maps $\phi_a : V_a \to \mathbb{R}^n$, $\phi_b : V_b \to \mathbb{R}^n$ and $\phi_{ab} : V_{ab} \to \mathbb{R}^n$, where V_a, V_b, V_{ab} are neighborhoods around a, b, ab that admit such a map. To see this visually, look at Figure 1 on the next page. We say that the group law is smooth if $\forall a, b \in G$ and $\forall \boldsymbol{x} \in \mathrm{Im}(V_a), \forall \boldsymbol{y} \in \mathrm{Im}(V_b)$,

$$\boldsymbol{\psi}_{ab}\left(\boldsymbol{x}, \boldsymbol{y}\right) := \phi_{ab}(\phi_a^{-1}(\boldsymbol{x})\phi_b^{-1}(\boldsymbol{y})) \tag{1}$$

is a smooth map. Since this is a function from $\mathbb{R}^{2n} \to \mathbb{R}^n$ smoothness is a well-defined concept, and one could write various explicit definitions, depending on the need. Next, we say that the group law inversion is smooth if $\forall a \in G$,

$$\boldsymbol{\psi}_a(\boldsymbol{x}) := \phi_{a^{-1}}\left(\left(\phi_a^{-1}(\boldsymbol{x})\right)^{-1}\right) \tag{2}$$

is a smooth map. Be aware that $\phi_a^{-1}(\boldsymbol{x}) \in V_a \subset G$, and therefore there are two kinds of inversion at play in (2). For a further discussion of these definitions, I once again refer to Figure 1.

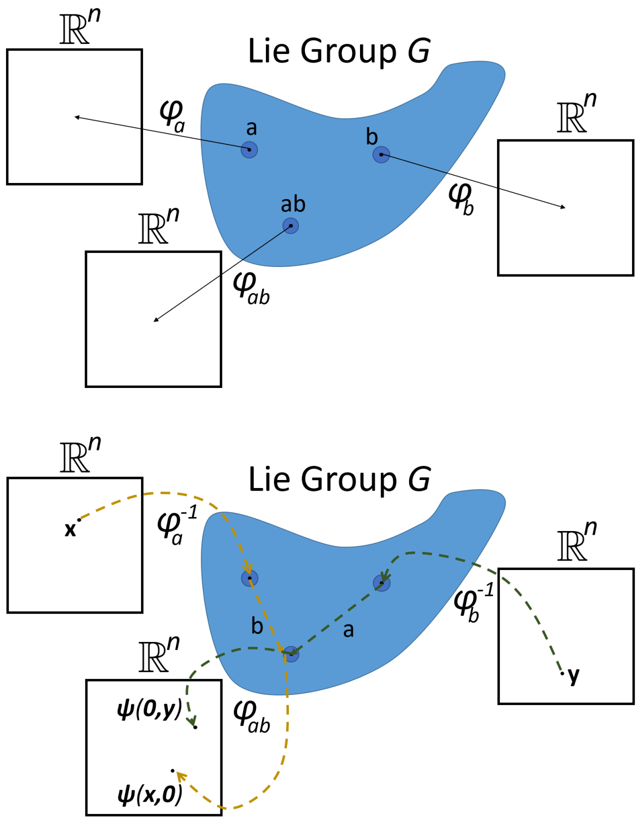

Figure 1: The top picture shows three different elements of the Lie Group G, their corresponding neighborhoods (not labelled), and the maps to $\mathbb{R}^n$. This is the framework we use in the bottom picture. Here I have sketched the path of the maps $\boldsymbol{\psi}_{ab}(0, \boldsymbol{y})$, $\boldsymbol{\psi}_{ab}(\boldsymbol{x}, 0)$. For the mapping to make sense, it is seen that multiplying nearby elements of the group with a third must give results in the neighborhood of each other, as our intuition also suggests of smooth maps.

3 Exercise 3.3.2: Proof that the identity component of a Lie Group G is a normal subgroup

Let G be a Lie Group, and let G_0 be the set of all $a \in G$ that are path-connected to e. Consider a specific point $a \in G_0$ and any $b \in G$. We know there exists a continuous map

$$f : [0, 1] \to G \text{ where } f(0) = e \text{ and } f(1) = a. \tag{3}$$

Consider now $g(t) := bf(t)b^{-1}$. Since G is a Lie Group, the group law is smooth, so $g(t)$ is also continuous. Thus $g(t)$ is a path on G. We know $g(0) = beb^{-1} = e$ and $g(1) = bab^{-1}$, so $g(t)$ path-connects e and bab^{-1}. Therefore we have shown $a \in G_0 \implies bab^{-1} \in G_0$, so G_0 is a normal subgroup of G.

References

[1] Richard Serge. *Groups and their representations.* 2023.