On Compactness

Exercise 3.2.4

On $\mathbb{R}$ show that any closed interval is a compact set, while any open interval is not compact. More generally, show that any closed and bounded set in $\mathbb{R}^n$ is compact.

Statement 1. *Any closed interval in $\mathbb{R}$ is a compact set.*

Proof. We set $I := [a, b]$ to be any closed interval in $\mathbb{R}$ and let $\mathcal{U}$ be an arbitrary open cover of I ($I \subset \mathcal{U}$). Clearly, for any $x \in I, \mathcal{U}$ is also an open cover of $[a, x]$. Let, for some $x \in [a, b]$, $[a, x]$ take up a finite subcover. Then we set $I' = \{x \in I \,|\, \mathcal{U} \text{ includes finite coverings of } [a, x]\}$. Notice that we only have to prove that $b \in I'$. Clearly, $I' \neq \phi$ because $a \in I'$. Also, we know that b is the upperbound of I'. Now we set $c := \max I'$, $c \leq b$.
Since, $c \in I'$ $\exists\, U \in \mathcal{U}$ such that $c \in U$ or for an ϵ small enough and $\epsilon > 0$

$$[c - \epsilon, c + \epsilon] \subset U \qquad \text{(because } U \text{ is an open subcover.)}$$

Let's prove $c = b$. Assume that $c < b$ then by choosing ϵ small enough,

$$c + \epsilon < b$$
$$\implies c + \epsilon \in I$$

and since, U and $\mathcal{U}$ cover $c + \epsilon$,

$$\implies c + \epsilon \in I'$$

However, this contradicts the fact that $c = \max I'$. Hence, our assumption was wrong and $c = b$.

$$\implies b \in I' \implies [a, b] \text{ is compact.}$$

□

An alternate proof of Statement 1.

Proof. Assume $I = [a, b]$ is not compact. We can also write $I = \left[a, \frac{a+b}{2}\right] \bigcup \left[\frac{a+b}{2}, b\right]$. Then, at least either of them is not compact. So we set $I_1 = [a_1, b_1]$ to be the one which is not compact. But now, I_1 can be written as $I_1 = \left[a_1, \frac{a_1+b_1}{2}\right] \bigcup \left[\frac{a_1+b_1}{2}, b_1\right]$. Since, I_1 is not compact, either one of $\left[a_1, \frac{a_1+b_1}{2}\right]$, $\left[\frac{a_1+b_1}{2}, b_1\right]$ must not be compact. So we set I_2 to be the one which is not compact. Following the same procedure for $I_2, I_3 \ldots I_{n-1}$ we can choose a non compact interval $I_n = [a_n, b_n]$. Notice that the sequence $\{I_k\}_{k=1}^n$ satisfies these conditions:

1. $I_1 \supset I_2 \supset \cdots \supset I_n \supset \ldots$
2. $\lim_{n\to\infty}(b_n - a_n) = 0$

Therefore, we conclude that $\lim_{n\to\infty} I_n = \{\alpha\}$, with $\alpha \in [a, b]$ by Nested intervals theorem.

Since I_n is not compact $\implies$ $\{\alpha\}$ is not compact. Which is not true, since a singleton set is always compact. Hence, our assumption was wrong and $[a, b]$ is compact. □

Statement 2. *Any closed and bounded set in $\mathbb{R}^n$ is compact.*

To prove this statement, we start by proving the following lemma.

Lemma 1. *If $S_1, S_2, S_3 \dots S_n$ are compact sets of $\mathbb{R}$, then $\mathbf{S} = S_1 \times S_2 \times \dots S_n$ is also a compact set in $\mathbb{R}^n$*

Proof. We start with an inductive approach. Let's start by proving this lemma for two compact sets A and B. Let $\{O_\lambda\}_{\lambda \in \Lambda}$ be an open cover of $A \times B$. For each $(a, b) \in A \times B$, $\exists \lambda(a, b) \in \Lambda$ such that $(a, b) \in O_{\lambda(a,b)}$ Since, $O_{\lambda(a,b)}$ is open, the point (a, b) is included in some open box say $X = U_{(a,b) \times V_{(a,b)}} \subset O_{\lambda(a,b)}$ where, $U_{(a,b)} \subset A$ and $V_{(a,b)} \subset B$.
Suppose we fix an a and vary b, then for every point (a, b) we find that the point is contained in an open box in $A \times B$. Proceeding in this manner, we observe that the collection of sets $\{V_{(a,b)}\}_{b \in B}$ is an open cover of B. Since B is compact, we can find a finite subcover $V_{(a,b_j(a))}$ of B containing the points $\{(a, b_j(a))\}$.
Now let $U_a = \bigcap_j U_{(a,b_j(a))}$. Since U_a is an intersection of finitely many open sets, it is itself open. Additionally, since A is compact, there are finitely many a_i such that U_{a_i} forms and open cover of A. Then it follows that the collection of sets $\left\{O_{(a_i,b_i(a)}\right\}$ (for all combinations of i, j) is a finite cover of $A \times B$, which implies $A \times B$ is compact. Hence by induction,

$$S_1 \times S_2 \times \dots S_n \text{ is compact.}$$

□

Notice that **Lemma 1** allows us to generate a n *dimensional* cube in $\mathbb{R}^n$.

Proof. Let X be an arbitrary, closed and bounded subset of $\mathbb{R}^n$. Let $\mathcal{U}$ be an arbitrary open cover of X. Let's assume that $\mathcal{U}$ has no finite subcover of X. Since X is bounded, we can say that it is contained within a *n-dimensional cube* of sufficient size, let's call this cube. $\mathcal{C}$.

$$\implies X \subset \mathcal{C}$$

Notice that since X is closed, $(\mathbb{R}^n - X)$ is an open set in $\mathbb{R}^n$, by definition. Also, $\mathcal{U} \cup (\mathbb{R}^n - X)$ forms an open cover of $\mathcal{C}$. According to our assumption that $\mathcal{U}$ does not have a finite subcover of X,

$$\implies \mathcal{U} \cup (\mathbb{R}^n - X) \text{ has no finite subcover of } \mathcal{C}.$$

But by **Lemma 1** we know that $\mathcal{C}$ is compact and therefore every open cover of $\mathcal{C}$ must have a finite subcover. Hence, our assumption was wrong and $\mathcal{U}$ contains a finite subcover of X

$$\therefore X \text{ is compact}$$

□