Restricted Lorentz Group and Lorentz Invariance in Physics

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Preliminary Proofs

Before studying the restricted Lorentz Group, let us prove that any element Λ of the Lorentz group verifies $\mathrm{Det}(\Lambda) = \pm 1$, and also $|\Lambda^0_{\ 0}| \geq 1$, with $\Lambda^0_{\ 0}$ is defined as the first entry of the matrix Λ. Afterwards, the proof of the Lorentz group can be divided into four disjoint groups will be provided. Any element $\Lambda \in \mathcal{L}$ satisfies the following relation $\Lambda^T g \Lambda = g$ with $g = diag(1, -1, -1, -1)$ and $g^2 = \mathbb{1} \iff g^{-1} = g$. Thus, by using the relation $\Lambda^T g \Lambda = g$, one has

$$\mathrm{Det}(\Lambda^T g \Lambda) = \mathrm{Det}(\Lambda^T)\mathrm{Det}(g)\mathrm{Det}(\Lambda) = \mathrm{Det}(g).$$

But we know that $\mathrm{Det}(\Lambda^T) = \mathrm{Det}(\Lambda)$. Therefore,

$$(\mathrm{Det}(\Lambda))^2 = 1 \implies \mathrm{Det}(\Lambda) = \pm 1.$$

Let us rewrite the relation $\Lambda^T g \Lambda = g$ into tensor-index notation, namely

$$g_{\mu\nu} \Lambda^\mu_{\ \alpha} \Lambda^\nu_{\ \beta} = g_{\alpha\beta}.$$

Consider the $(\alpha, \beta) = (0, 0)$ component

$$g_{\mu\nu} \Lambda^\mu_{\ 0} \Lambda^\nu_{\ 0} = g_{00} = 1 \iff \Lambda_{\nu 0} \Lambda^\nu_{\ 0} = \Lambda_0 \cdot \Lambda_0 = \left(\Lambda^0_{\ 0}\right)^2 - (\Lambda^i_{\ 0})^2 = 1 \iff \left(\Lambda^0_{\ 0}\right)^2 = 1 + (\Lambda^i_{\ 0})^2 \geq 1$$

where the summation over repeated indices is understood (Einstein summation convention) and we have used $g_{\mu\nu} \Lambda^\mu_{\ \alpha} = \Lambda_{\nu\alpha}$ and the Minkowski inner product in tensor-index notation, namely $A \cdot B := A_\nu B^\nu = A^\mu g_{\mu\nu} B^\nu = A^0 B^0 - A^i B^i$. Hence, we have $|\Lambda^0_{\ 0}| \geq 1$. Moreover, we know that Λ has an inverse Λ^{-1} since $\det(\Lambda) \neq 0$ and in fact, the inverse also preserves the bilinear map, namely

$$g = (\Lambda^T)^{-1} \Lambda^T g \Lambda \Lambda^{-1} = (\Lambda^T)^{-1} g \Lambda^{-1} = (\Lambda^{-1})^T g \Lambda^{-1}.$$

Therefore, $\Lambda^{-1} \in \mathcal{L}$ and if we take the inverse of the relation $\Lambda^T g \Lambda = g$, one has

$$\Lambda^{-1} g (\Lambda^{-1})^T = g \iff \Lambda g \Lambda^T = g$$

where the property $g^{-1} = g$ has been used in the expression. Consequently, if one expresses the inverse relation in tensor-index notation and one takes the $(0, 0)$ component, one has

$$\Lambda_{0\nu} \Lambda^0_{\ \nu} = \Lambda^0 \cdot \Lambda^0 = \left(\Lambda^0_{\ 0}\right)^2 - (\Lambda^0_{\ i})^2 = 1 \iff \left(\Lambda^0_{\ 0}\right)^2 = 1 + (\Lambda^0_{\ i})^2 \geq 1.$$

Consider $\Lambda, \Lambda' \in \mathcal{L}$. The matrix multiplication yields

$$(\Lambda\Lambda')^0_{\ 0} = \Lambda^0_{\ 0}\Lambda'^0_{\ 0} + \Lambda^0_{\ k}\Lambda'^k_{\ 0} = \left(1 + (\Lambda^0_{\ i})^2\right)^{1/2}\left(1 + (\Lambda'^j_{\ 0})^2\right)^{1/2} + \Lambda^0_{\ k}\Lambda'^k_{\ 0}.$$

Using the Cauchy-Schwarz inequality, one infers that

$$\left|\sum_{k=1}^{3}\Lambda^0_{\ k}\Lambda'^k_{\ 0}\right| \leq \left(\sum_{i=1}^{3}(\Lambda^0_{\ i})^2\right)^{1/2}\left(\sum_{j=1}^{3}(\Lambda'^j_{\ 0})^2\right)^{1/2}.$$

Let us now fix x, y such that $(\Lambda^0_{\ i})^2 = \sinh^2(x)$ and $(\Lambda'^j_{\ 0})^2 = \sinh^2(y)$ where the summation over repeated indices is understood. If $\Lambda^0_{\ 0}$ and $\Lambda'^0_{\ 0}$ have the same sign, then one has

$$\begin{aligned}
(\Lambda\Lambda')^0_{\ 0} &\geq \left(1 + \sum_{i=1}^{3}(\Lambda^0_{\ i})^2\right)^{1/2}\left(1 + \sum_{j=1}^{3}(\Lambda'^j_{\ 0})^2\right)^{1/2} - \left(\sum_{i=1}^{3}(\Lambda^0_{\ i})^2\right)^{1/2}\left(\sum_{j=1}^{3}(\Lambda'^j_{\ 0})^2\right)^{1/2} \\
&= \cosh(x)\cosh(y) - \sinh(x)\sinh(y) \\
&= \cosh(x-y) \\
&\geq 1.
\end{aligned}$$

If $\Lambda^0_{\ 0}$ and $\Lambda'^0_{\ 0}$ have the opposite sign, then by using the same trick as before, one has

$$\begin{aligned}
(\Lambda\Lambda')^0_{\ 0} &\leq -\left(1 + \sum_{i=1}^{3}(\Lambda^0_{\ i})^2\right)^{1/2}\left(1 + \sum_{j=1}^{3}(\Lambda'^j_{\ 0})^2\right)^{1/2} + \left(\sum_{i=1}^{3}(\Lambda^0_{\ i})^2\right)^{1/2}\left(\sum_{j=1}^{3}(\Lambda'^j_{\ 0})^2\right)^{1/2} \\
&= -\cosh(x)\cosh(y) + \sinh(x)\sinh(y) \\
&= -\cosh(x-y) \\
&\leq -1.
\end{aligned}$$

From the results above, one infers that $|(\Lambda\Lambda')^0_{\ 0}| \geq 1$ for any $\Lambda\Lambda' \in \mathcal{L}$ and the components determined by $\Lambda^0_{\ 0}$ are disjoint. Then, let us define the restricted Lorentz group as follows

Definition 1

The restricted Lorentz group $\mathcal{L}^{\uparrow}_{+}$ is defined as the Lorentz group $\mathcal{L}$ that is proper and orthochronous, namely one has ([2])

$$\mathcal{L}^{\uparrow}_{+} := \{\Lambda \in \mathcal{L} \mid \mathrm{Det}(\Lambda) = 1 \text{ and } \Lambda^0_{\ 0} \geq 1\}.$$

Furthermore, the restricted Lorentz group can be denoted by $\mathrm{SO}^{+}(1,3)$ where $(1,3)$ is the signature of the quadrature form and the "+" denotes the orthochronous property of $\mathcal{L}^{\uparrow}_{+}$.

Proper and orthochronous Lorentz transformations

For $\Lambda \in \mathcal{L}_+^\uparrow$, the restricted Lorentz transformation (proper and orthochronous) is denoted by ([4])

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu.$$

Moreover, one needs to keep the 4-vector inner product invariant. Suppose A^μ and B^μ are transformed by the same matrix Λ. Namely,

$$A'^\mu = \Lambda^\mu{}_\alpha A^\alpha, \; B'^\nu = \Lambda^\nu{}_\beta B^\beta.$$

Then, let us consider the 4-vector inner product

$$\begin{aligned} A' \cdot B' = A'_\nu B'^\nu = g_{\mu\nu} A'^\mu B'^\nu = (g_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta) A^\alpha B^\beta. \\ A \cdot B = A_\beta B^\beta = g_{\alpha\beta} A^\alpha B^\beta. \end{aligned}$$

Therefore, the condition such that the 4-vector inner product invariant, namely the equality $A' \cdot B' = A \cdot B$ holds, is given by

$$g_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\alpha\beta}$$

Observe that the relationship above is equivalent to the relation which has been written in the first section, namely $\Lambda^T g \Lambda = g$ (The Lorentz group preserves the bilinear map).

Proper Rotations

A restricted Lorentz transformation $\Lambda \in \mathcal{L}_+^\uparrow$ is said to be proper rotation if it leaves the time unchanged, namely $\Lambda^0{}_0 = 1$. Then, the pure rotation has the following form ([1])

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{R} \end{pmatrix}$$

with $\mathcal{R}$ denotes the three-dimensional rotation part of Λ with $\mathcal{R} \in SO(3)$. For a rotation about some vector $\vec{n}$ in 3-space, the rotation leaves $\vec{n}$ unchanged and acts in the plane orthogonal to $\vec{n}$. For example, consider the rotation about the third axis $\vec{n} = \vec{e}_3$ and if we express $\mathcal{R}$ altogether with $\Lambda_{00} = 1$, then the pure rotation has the following form

$$\Lambda(\vec{e}_3, \theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

One can interpret the rotation as rotating the coordinate system or rotating the space in a fixed coordinate system depending on the sign of θ. The former is called a passive transformation and the latter is called an active transformation. Let us check if Λ preserves the bilinear map

$$
\begin{aligned}
\Lambda^T g \Lambda &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
&= g.
\end{aligned}
$$

Hence, Λ preserves the bilinear map. Moreover, one also has $\det(\Lambda) = 1$ and $\Lambda^0_{\ 0} = 1$ such that proper rotation transformation is restricted Lorentz transformation.

Pure Lorentz Boosts

A restricted Lorentz transformation $\Lambda \in \mathcal{L}_+^\uparrow$ is said to be a pure boost in the direction of a certain 3-space vector $\vec{n}$ if it leaves unchanged any vectors in 3-space in the plane orthogonal to $\vec{n}$. Then, there exists another parameter η which determines the magnitude of the boost. By choosing the 3-space vector as $\pm\vec{n}$, then we have $\eta \geq 0$. For example, the pure Lorentz boost along the first coordinate axis can be represented by the following matrix

$$
\Lambda(\vec{e}_1, \eta) = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

One can observe that $\det(\Lambda) = 1$ and $\Lambda_{00} = \cosh\eta \geq 1$ which agrees with our definition of proper and orthochronous Lorentz transformation. Then, let us check if the 4-vector inner product is invariant, namely we check the following condition $\Lambda^T g \Lambda = g$

$$\begin{aligned}
\Lambda^T g \Lambda &= \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ \sinh\eta & -\cosh\eta & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
&= g
\end{aligned}$$

Hence, the pure boost transformation is proper and orthochronous Lorentz transformation ($\Lambda \in \mathcal{L}_+^\uparrow$).

Lorentz Invariance

Let us consider a scalar field ϕ under the Lorentz transformation $x \to \Lambda x$, namely

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$$

The inverse Λ^{-1} appears in the argument because we consider an active transformation in which the field is truly shifted. The definition of a Lorentz invariant theory is that if ϕ solves the equations of motion then $\phi(\Lambda^{-1}\cdot)$ also solves the equations of motion. Meaning that the laws of physics **are the same** for different observers even though the frame of reference is rotated through some angle or traveling at a constant speed relative to the observer at rest. We can ensure that this property holds by requiring that the action is Lorentz invariant ([3]). Let us consider a famous example in relativistic quantum mechanics,

The Klein-Gordon Equation

Consider the Lagrangian for a real scalar field $\phi(\vec{x}, t)$ ([3]),

$$\mathcal{L}(\phi) = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2$$

This real scalar field has been transformed under Lorentz transformation, $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$. The derivative of the scalar field transforms as a vector, namely

$$(\partial_\mu \phi)(x) \to (\Lambda^{-1})^\nu{}_\mu (\partial_\nu \phi)(y)$$

with $y = \Lambda^{-1}x$. Here, the potential terms transform in the following way $\phi^2(x) \to \phi^2(y)$ meaning that the potential terms are invariant under the transformation. Consider the derivative terms of the Lagrangian

$$\begin{aligned}\mathcal{L}_{deriv}(x) = \partial_\mu\phi(x)\partial_\nu\phi(x)g^{\mu\nu} &\to (\Lambda^{-1})^{\alpha}{}_{\mu}(\partial_\alpha\phi)(y)(\Lambda^{-1})^{\beta}{}_{\nu}(\partial_\beta\phi)(y)g^{\mu\nu} \\ &= (\partial_\alpha\phi)(y)(\partial_\beta\phi)(y)g^{\alpha\beta} \\ &= \mathcal{L}_{deriv}(y)\end{aligned}$$

Therefore, the action is given by

$$S = \int d^4x\mathcal{L}(x) \to \int d^4x\mathcal{L}(y) = \int d^4y\mathcal{L}(y) = S$$

From this result, one infers that the action is invariant under **proper** Lorentz transformations (since we have $\det(\Lambda) = 1$, then we don't need to take into account the Jacobian factor).

References

[1] Arthur Jaffe. *Lorentz Transformations, Rotations, and Boosts.* 2015.

[2] Serge Richard. *Special Mathematics Lecture: Groups and their representations.* 2022.

[3] David Tong. *Lectures on Quantum Field Theory.* 2006.

[4] Hitoshi Yamamoto. *Quantum Field Theory for Non-Specialists (Lecture Notes).*