# $Z(G)$ is an Abelian and Normal subgroup of $G$

FIRDAUS Rafi Rizqy / 062101889

Special Mathematics Lecture: Groups and their representations (Fall 2022)

### Exercise 1.2.12

As defined in **Definition 1.2.11**, the center of the group $G$ is defined by $Z(G) : \{a \in G \mid ab = ba \; \forall \, b \in G\}$. We want to prove that $Z(G)$ is a subgroup of $G$ by checking the following properties of a subgroup.

1) The operation of $Z(G)$ is associative as the operation of $Z(G)$ is the same as the operation of the group $G$ and $G$ itself is a group which satisfies the associativity property ($\forall a, b, c \in G : (ab)c = a(bc)$).

2) The identity element $e$ is also in the center of group $G$ ($e \in Z(G)$) as $ea = ae = a$ for any $a \in G$.

3) We want to check if $Z(G)$ is closed under the operation of $G$. Let $a, b \in Z(G)$, then $\forall c \in G$ one has

$$\begin{aligned}(ab)c &= a(bc)\\ &= a(cb)\\ &= (ac)b\\ &= (ca)b\\ &= c(ab).\end{aligned}$$

Thus, $ab \in Z(G)$ and therefore $Z(G)$ is closed under the operation of $G$.

4) We want to check if the inverse is in $Z(G)$. Let $a \in Z(G)$, then $\forall b \in G$ one has

$$\begin{aligned}a^{-1}b &= a^{-1}b(aa^{-1})\\ &= a^{-1}(ba)a^{-1}\\ &= a^{-1}(ab)a^{-1}\\ &= ba^{-1}.\end{aligned}$$

Thus, $a^{-1} \in Z(G)$ and the inverse exists for every element in $Z(G)$.

$Z(G)$ is, therefore, a subgroup of $G$ and by its definition, we also know that $Z(G)$ is an Abelian subgroup of $G$. Then, we want to show that $Z(G)$ is a normal subgroup. Let $a \in Z(G)$, then $\forall b \in G$ one can write

$$\begin{aligned} a &= a(bb^{-1}) \\ &= (ab)b^{-1} \\ &= bab^{-1}. \end{aligned}$$

Hence, we have $bZ(G)b^{-1} = Z(G)$, $\forall b \in G$ which implies that $Z(G)$ is also a normal subgroup. $Z(G)$ **is, therefore, an abelian and normal subgroup of** $G$.