The Restricted Lorentz Group

and Notions of Orthochronous Proper Lorentz Transformations

(1.5.4)

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We recall the Lorentz group $\mathcal{L}$ consists of the set of all $\mathbb{R}^4 \times \mathbb{R}^4$ matrices $\Lambda \in M_4(\mathbb{R})$ such that

$$(\Lambda x) \cdot (\Lambda \mathrm{y}) = x \ \cdot y \qquad \text{for any } x, y \in \mathbb{M} \text{ (the Minkowski Space)}$$

with a matrix form

$$\Lambda^T g \Lambda = \mathrm{g}$$

as shown in Section 1.5 of the lecture notes.

As introduced in Section 5.5 of the lecture notes, the Lorentz Group $\mathcal{L}$ is a six-dimensional subgroup of the Poincare Group, and can be divided into four non-simply connected components:

$$\mathcal{L}_+^\uparrow := \{\Lambda \in \mathcal{L} \mid Det(\Lambda) = 1 \ and \ \Lambda_0^0 \geq 1\}$$

$$\mathcal{L} := \{\Lambda \in \mathcal{L} \mid Det(\Lambda) = -1 \ and \ \Lambda_0^0 \geq 1\}$$

$$\mathcal{L}_+^\downarrow := \{\Lambda \in \mathcal{L} \mid Det(\Lambda) = 1 \ and \ \Lambda_0^0 \leq -1\}$$

$$\mathcal{L}_-^\downarrow := \{\Lambda \in \mathcal{L} \mid Det(\Lambda) = -1 \ and \ \Lambda_0^0 \leq -1\}$$

$\mathcal{L}_+^\uparrow$ is the called the Restricted Lorentz Group as it is continuously connected to the identity component *e* of the Lorentz Group $\mathcal{L}$.

Furthermore, the elements of the union $\mathcal{L}^\uparrow$ of $\mathcal{L}_+^\uparrow$ and $\mathcal{L}_-^\uparrow$ are considered Orthochronous, as they preserve the direction of time within the Minkowski Space (represented by the first entry of a vector). We can see this when we apply an Orthochronous transformation of $\mathcal{L}^\uparrow$ to an element of the Minkowski space (an element of $\mathbb{R}^4$). Since the first entry of Λ for the elements of $\mathcal{L}^\uparrow$ is greater than or equal to one, the first entry's direction of Λx is preserved. It follows that since the first entry of Λ in $\mathcal{L}^\downarrow$ is less than or equal to negative one, the direction of time would be altered.

Remarks: Orthochronous Lorentz transforms are also called Proper Lorentz Transforms, while the remaining union of $\mathcal{L}^\downarrow$ components is referred to as Improper Lorentz Transforms.

See here (Wikipedia) for additional information.