Reminder VI

- Tensor product of 2 Hilbert spaces : $\mathcal{H}_1 \otimes \mathcal{H}_2$ has basis $\{e^1_j \otimes e^2_k\}_{j,k}$ if $\{e^1_j\}_j$ is a basis of $\mathcal{H}_1$, $\{e^2_k\}_k$ is a basis of $\mathcal{H}_2$.

- If $A_j \in \mathcal{B}(\mathcal{H}_j)$, $A_1 \otimes A_2 \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ acts as $(A_1 \otimes A_2)(f_1 \otimes f_2) = A_1 f_1 \otimes A_2 f_2$.

- $(\mathcal{H}_1, U_1)$ rep. of G_1, $(\mathcal{H}_2, U_2)$ rep. of G_2, then $(\mathcal{H}_1 \otimes \mathcal{H}_2, U_1 \otimes U_2)$ rep. of $\underbrace{G_1 \times G_2}_{\text{direct product}}$. If both initial rep. are irreducible, tensor product representation is irreducible.

- $(\mathcal{H}_1, U_1)$, $(\mathcal{H}_2, U_2)$ rep. of G, then $(\mathcal{H}_1 \otimes \mathcal{H}_2, U_1 \otimes U_2)$ representation of G, usually reducible $\Rightarrow$

$(\mathcal{H}^\ell, U^\ell)$ irreducible rep.

$(\mathcal{H}_1 \otimes \mathcal{H}_2, U_1 \otimes U_2) = (\oplus_\ell \nu_\ell \mathcal{H}^\ell, \oplus_\ell \nu_\ell U^\ell)$. If

irreducible

$(\mathcal{H}_1, U_1) = (\mathcal{H}^j, U^j)$, $(\mathcal{H}_2, U_2) = (\mathcal{H}^k, U^k)$, then

2 natural bases for $\mathcal{H}^j \otimes \mathcal{H}^k$ $\rightsquigarrow$ Clebsch - Gordan coefficients.

- One more step of abstraction ⇝ selection rule.
- Projective Hilbert space $\hat{\mathcal{H}} = \mathcal{H}/\mathbb{C}$, $\hat{\varphi} \in \hat{\mathcal{H}}$ if $\exists\, \varphi \in \mathcal{H}$, $\|\varphi\| = 1$ and $\hat{\varphi} = \{\lambda\varphi \mid \lambda \in \mathbb{C}\}$. Bijection with pure states: $|\varphi\rangle\langle\varphi| \in \mathcal{B}(\mathcal{H})$, $\|\varphi\| = 1$ (orthogonal projection ($P = P^* = P^2$)). If $\hat{\varphi} \in \hat{\mathcal{H}}$, set $P_{\hat{\varphi}} := |\varphi\rangle\langle\varphi|$.
- Transition probability: $\mathrm{Tr}(P_{\hat{\varphi}} P_{\hat{g}}) = |\langle \varphi, g\rangle|^2 \quad \forall \hat{\varphi}, \hat{g} \in \hat{\mathcal{H}}$
- Symmetries: $S : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ s.t. $\mathrm{Tr}(P_{S\hat{\varphi}} P_{S\hat{g}}) = \mathrm{Tr}(P_{\hat{\varphi}} P_{\hat{g}})$.