Reminder V (representation of finite groups)

- $\{\gamma^k\}_k$ set of equivalence classes of inequivalent irreducible representations, $(\mathcal{H}^k, U^k) \in \gamma^k$ a unitary representation, $\{e_j^k\}_{j=1}^{n_k}$ an orthonormal basis of $\mathcal{H}^k$. Set

of dim. n_k

$U_{ij}^k(a) := \langle e_i^k, U^k(a) e_j^k \rangle$, then

orthogonality relations in $\mathbb{C}^{|G|}$

$$\frac{1}{|G|} \sum_{a\in G} U^\ell_{rs}(a) \overline{U^k_{ij}(a)} = \frac{1}{n_k} \delta_{k\ell} \delta_{sj} \delta_{ri} .$$

- $\leadsto$ $\sum_k n_k \leq |G|$. It is in fact an equality

- Character : $(\mathcal{H}, U)$ finite dim representation,

$$\chi_U(a) := \mathrm{Tr}(U(a)). \qquad \chi_U : G \to \mathbb{C} \iff \chi_U \in \mathbb{C}^{|G|}.$$

- Decomposition : $(\mathcal{H}, U) = (\oplus_k \nu_k \mathcal{H}^k, \oplus_k \nu_k U^k)$,

irreducible

then $\nu_k = \frac{1}{|G|} \sum_{a\in G} \chi_U(a) \overline{\chi^k(a)}$.

$(\mathcal{H}, U)$ is irreducible iff $\frac{1}{|G|} \sum |\chi_U(a)|^2 = 1$.

- Regular representation : $\mathcal{H}^{reg} := \ell^2(G)$ functions from G to $\mathbb{C}$.

$[U^{reg}(a) f](b) = f(a^{-1} b)$. $(\mathcal{H}^{reg}, U^{reg})$ is a unitary rep., containing each irreducible representations

$(\mathcal{H}^k, U^k)$ $\quad n_k$ times $\quad (\nu_k = n_k)$

$$\rightsquigarrow \quad \sum_k n_k^2 = |G| .$$

- The number of conjugacy classes in G is equal to the number of inequivalent irreducible rep. of G.