Reminder IV

- Linear representation $U : G \to \mathcal{L}(\mathcal{V})$, $U(ab) = U(a)U(b)$
  $U(e) = 1\!\!1$.

Hilbert space

For $U : G \to \mathcal{B}(\mathcal{H})$, if $U(a)$ is unitary $\forall a \in G$,
we call it a unitary representation.

equivalent ⇝ define equivalence relation

- $(\mathcal{V}, U) \simeq (\mathcal{V}', U')$ if $\exists$ bijective $T : \mathcal{V} \to \mathcal{V}'$ s.t.
  $U'(a) = T\,U(a)\,T^{-1}$ $\forall a \in G$. If $T : \mathcal{H} \to \mathcal{H}'$
  is unitary, the representations are unitarily equivalent.
- For G finite, $(\mathcal{H}, U)$ is always equivalent to a unitary rep.
- Subspace, complementary subspace, orthogonal complement.
  $T = \begin{pmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{pmatrix}$ in $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$.
- Invariant subspace, minimal subspace, irreducible rep.
- All irreducible representations of a finite group are of dim $\leq |G|$.
- If $|G| < \infty$ and $\dim(\mathcal{H}) < \infty$, then any unitary representation
  decomposes into $\mathcal{H} = \oplus_k \mathcal{H}^k$, $U = \oplus_k U^k$,
  with $(\mathcal{H}^k, U^k)$ irreducible.

<u>Schur's lemma</u> : $(\mathcal{V}, U)$, $(\mathcal{V}', U')$ 2 irreducible representations of G, and $T : \mathcal{V} \to \mathcal{V}'$ linear and satisfying $T U(a) = U'(a) T$ $\forall a \in G$. Then either $T = 0$, or $(\mathcal{V}', U') \simeq (\mathcal{V}, U)$ and T bijective.

<u>Corollary</u> : If $(\mathcal{V}, U)$ finite dim. and irreducible rep. of G, and $T U(a) = U(a) T$ $\forall a \in G$, then $T = \lambda \mathbb{1}$ for some $\lambda \in \mathbb{C}$.

<u>Corollary</u> : Any irreducible representation of an Abelian group is 1 dimensional.