
Homework 6

Exercise 1 Find the critical points for the differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ defined for $x \in \mathbb{R}$ by

$$a) -x^2 + 2x + 2, \quad b) x^3 - 3, \quad c) \cos(x), \quad d) \sin(x) + \cos(x).$$

Exercise 2 Recall that $e^{-x} := \frac{1}{e^x}$, and consider the functions hyperbolic cosine $\cosh : \mathbb{R} \rightarrow \mathbb{R}$ and hyperbolic sine $\sinh : \mathbb{R} \rightarrow \mathbb{R}$ defined by the formulas

$$\cosh(x) := \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh(x) := \frac{e^x - e^{-x}}{2}.$$

Compute the derivative of these functions and sketch the graph of the functions \cosh and \sinh . Prove the following relation:

$$\cosh(x)^2 - \sinh(x)^2 = 1, \quad \forall x \in \mathbb{R}.$$

Exercise 3 Find the point of the curve of equation $y^2 = 4x$ which is the nearest one to the point $(2, 3)$.

Exercise 4 Show that $\sin(x) \leq x$ for any $x \geq 0$.

Exercise 5 Show that there are exactly two tangent lines to the graph of the function $f : \mathbb{R} \ni x \mapsto (x+1)^2 \in \mathbb{R}$ which pass through the origin. Find the equation of these lines (\Leftrightarrow find the two functions whose graphs correspond to these straight lines).

Exercise 6 Prove the following statement: Let $f : [a, b] \rightarrow \mathbb{R}$ be a strictly increasing and continuous function, and set $\alpha := f(a)$ and $\beta := f(b)$. Then there exists an inverse function $f^{-1} : [\alpha, \beta] \rightarrow [a, b]$ such that $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$ for any $x \in [a, b]$ and $y \in [\alpha, \beta]$.

Exercise 7 (Midterm 2019) For any $x \in \mathbb{R}$ with $x \neq -1$ we consider the sequence $(a_n)_{n \in \mathbb{N}}$ given by

$$a_n := \frac{x^n - 1}{1 + x^n}.$$

For which x does the limit $a_\infty := \lim_{n \rightarrow \infty} a_n$ exist? Give the value of this limit whenever it exists. Represent your findings on a graph (the horizontal axis corresponds to the x -variable, the vertical axis to the values of a_∞).