Supplementary Note on Green's Function Method

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This document serves as a supplement to the Green's function method used in Guan Xinye's report. In Section 1, we shall give the definition of tempered distributions. In Section 2, we shall define a number of operations in the space of tempered distributions, including the multiplication (with a function), the weak derivative, the Fourier transform and the convolution. With these tools equipped, we will verify some arguments in Guan's report in Section 3.

Please be aware that the notations used in this note may be *different* from those in Guan's report.

I attempted to avoid using the results in advanced analysis, but it seems that not using the dominated convergence theorem in real analysis will make the proofs even longer. Indeed, an elementary approach to Fourier analysis is possible and actually presented in [1]. Since we will apply the dominated convergence theorem several times in this note, let me first state one weak version of it to make the arguments clear.

Theorem 0.1 (Dominated convergence theorem)**.** *Let* $(f_n)_{n=1}^{\infty}$ *be a sequence of complex-valued piecewise continuous functions on* $\mathbb{R}$*, and assume that this sequence converges pointwise to a piecewise continuous function* $f : \mathbb{R} \to \mathbb{C}$ *except at countably many points in* $\mathbb{R}$*. Then*

$$\lim_{n\to+\infty} \int_{-\infty}^{\infty} f_n(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} f(t)\,\mathrm{d}t,$$

provided that there exists a non-negative integrable function $g : \mathbb{R} \to \mathbb{R}$ *such that* $\sup_n |f_n(t)| \le g(t)$ *for all but countably many* $t \in \mathbb{R}$*.*

We shall always count natural numbers from 0. In other words, $0 \in \mathbb{N}$.

1 Tempered Distributions

Definition. *A* C^∞ *function* $f : \mathbb{R} \to \mathbb{C}$ *is called a* Schwartz function *if for any* $k, l \in \mathbb{N}$*,*

$$|f|_{k,l} \coloneqq \sup_{t\in\mathbb{R}} |t^k f^{(l)}(t)| < +\infty,$$

where $f^{(l)}$ denotes the l-th derivative of f. The set $\mathcal{S}(\mathbb{R})$ of all Schwartz functions is called the Schwartz space.

An example can be the Gaussian functions $t \mapsto \alpha \exp(-\beta t^2)$, for $\alpha \in \mathbb{R}$ and $\beta > 0$.

We also introduce a larger class of functions that will be used later.

$$BC^\infty(\mathbb{R}) \coloneqq \{\, f \in C^\infty(\mathbb{R}) \mid |f|_{0,l} < \infty, \forall l \in \mathbb{N} \,\}\,;$$
$$PBC^\infty(\mathbb{R}) \coloneqq \{\, pf \mid p \text{ is a polynomial, } f \in BC^\infty(\mathbb{R}) \,\}\,.$$

By definition, $\mathcal{S}(\mathbb{R}) \subseteq BC^\infty(\mathbb{R}) \subseteq PBC^\infty(\mathbb{R})$.

Proposition 1.1. *For any $c \in \mathbb{C}$, $f, g \in \mathcal{S}(\mathbb{R})$ and $h \in PBC^\infty(\mathbb{R})$, $cf + g$ and fh are Schwartz.*

The complete proof of Proposition 1.1 is not difficult yet rather tedious, so we shall omit the proof.

Proposition 1.2. *Every Schwartz function is absolutely integrable.*

Proof. Notice that for each Schwartz function f,

$$|f(t)| = \frac{(1+t^2)|f(t)|}{1+t^2} \leq \frac{|f|_{0,0} + |f|_{2,0}}{1+t^2}.$$

Therefore,

$$\int_{-\infty}^{\infty} |f(t)| \,\mathrm{d}t \leq \int_{-\infty}^{\infty} \frac{|f|_{0,0} + |f|_{2,0}}{1+t^2} \,\mathrm{d}t = \pi(|f|_{0,0} + |f|_{2,0}) < +\infty,$$

which shows the absolute integrability of f. □

Definition. *A* tempered distribution *on $\mathbb{R}$ is a linear functional $\phi : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ satisfying the continuity condition, which means that for any sequence $(f_n)_{n=1}^\infty$ of Schwartz functions and any $f \in \mathcal{S}(\mathbb{R})$,*

$$\lim_{n\to+\infty} \phi(f_n) = \phi(f)$$

whenever $\lim_{n\to+\infty} |f_n - f|_{k,l} = 0$ for any $k, l \in \mathbb{N}$. Denote by $\mathcal{S}'(\mathbb{R})$ the set of all tempered distributions on $\mathbb{R}$.

Example 1.3. Suppose that $f : \mathbb{R} \to \mathbb{C}$ is the product of a polynomial and a bounded piecewise continuous function. Define $f^* : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ by

$$f^*(g) \coloneqq \int_{-\infty}^{\infty} f(t) g(t) \,\mathrm{d}t$$

for any Schwartz function g. Then, f^* is a tempered distribution.

Proof. For f described above, let $m \in \mathbb{N}$, $a_0, \cdots, a_m \in \mathbb{C}$ and a bounded piecewise continuous $f_0 : \mathbb{R} \to \mathbb{C}$ satisfy

$$f(t) = \sum_{k=0}^{m} a_k t^k f_0(t)$$

for any $t \in \mathbb{R}$. By noting that

$$\begin{aligned}\int_{-\infty}^{\infty} |f(t)g(t)| \, \mathrm{d}t &= \int_{-\infty}^{\infty} \left| \sum_{k=0}^{m} a_k t^k f_0(t) g(t) \right| \mathrm{d}t \\ &\leq \sup_{t\in\mathbb{R}} |f_0(t)| \sum_{k=0}^{m} |a_k| \int_{-\infty}^{\infty} |t^k g(t)| \, \mathrm{d}t \\ &\leq \sup_{t\in\mathbb{R}} |f_0(t)| \sum_{k=0}^{m} |a_k| \int_{-\infty}^{\infty} \frac{|g|_{k,0} + |g|_{k+2,0}}{1+t^2} \, \mathrm{d}t \\ &= \pi \sup_{t\in\mathbb{R}} |f_0(t)| \sum_{k=0}^{m} |a_k| (|g|_{k,0} + |g|_{k+2,0}) < +\infty,\end{aligned}$$

we can see that $f^*(g)$ is well-defined for any $g \in \mathcal{S}(\mathbb{R})$.

The linearity of f^* follows immediately from the linearity of integration, so it only remains to show the continuity of f^*. Let $(g_n)_{n=1}^{\infty}$ be a sequence of Schwartz functions and $g \in \mathcal{S}(\mathbb{R})$. Assume that $\lim_{n\to+\infty} |g_n - g|_{k,l} = 0$ for any $k, l \in \mathbb{N}$. Then using the inequality derived before to show the well-definedness of $f^*(g)$, we obtain

$$\begin{aligned}0 \leq |f^*(g_n) - f^*(g)| &\leq \int_{-\infty}^{\infty} |f(t)(g_n(t) - g(t))| \, \mathrm{d}t \\ &\leq \pi \sup_{t\in\mathbb{R}} |f_0(t)| \sum_{k=0}^{m} |a_k| (|g_n - g|_{k,0} + |g_n - g|_{k+2,0}).\end{aligned}$$

As the last expression converges to 0 as n approaches infinity, the continuity of f^* holds. Summing up what we have deduced, we conclude that $f^* \in \mathcal{S}'(\mathbb{R})$. □

Lemma 1.4. *Let $f : \mathbb{R} \to \mathbb{C}$ be a bounded piecewise continuous function. Then, for any $g \in \mathcal{S}(\mathbb{R})$ and $t \in \mathbb{R}$, if f is continuous at t,*

$$\lim_{n\to+\infty} n \int_{-\infty}^{\infty} f(s) g(n(t-s)) \, \mathrm{d}s = f(t) \int_{-\infty}^{\infty} g(s) \, \mathrm{d}s.$$

Proof. By changing the variable, we have

$$n \int_{-\infty}^{\infty} f(s) g(n(t-s)) \, \mathrm{d}s = \int_{-\infty}^{\infty} f(t - s/n) g(s) \, \mathrm{d}s.$$

Notice that $|f(t-s/n)g(s)| \le |f|_{0,0} \cdot |g(s)|$ for any t and s, and g is absolutely integrable, so by the dominated convergence theorem,

$$\begin{aligned}\lim_{n\to+\infty} n\int_{-\infty}^{\infty} f(s)g(n(t-s))\,\mathrm{d}s &= \lim_{n\to+\infty}\int_{-\infty}^{\infty} f(t-s/n)g(s)\,\mathrm{d}s \\ &= \int_{-\infty}^{\infty}\lim_{n\to+\infty} f(t-s/n)g(s)\,\mathrm{d}s \\ &= f(t)\int_{-\infty}^{\infty} g(s)\,\mathrm{d}s,\end{aligned}$$

for any $t \in \mathbb{R}$ at which f is continuous. □

Proposition 1.5. *Suppose that $f : \mathbb{R} \to \mathbb{C}$ is a bounded function and $f^* = 0^*$. Then $f(t) = 0$ for any $t \in \mathbb{R}$ at which f is continuous.*

Proof. Arbitrarily pick one $g \in \mathcal{S}(\mathbb{R})$ satisfying $\int_{-\infty}^{\infty} g(s)\,\mathrm{d}s \ne 0$. Since $f^* = 0^*$, for any positive integer n,

$$n\int_{-\infty}^{\infty} f(s)g(n(t-s))\,\mathrm{d}s = 0.$$

Combining this with Lemma 1.4, we get

$$f(t)\int_{-\infty}^{\infty} g(s)\,\mathrm{d}s = \lim_{n\to+\infty} n\int_{-\infty}^{\infty} f(s)g(n(t-s))\,\mathrm{d}s = 0,$$

for any $t \in \mathbb{R}$ at which f is continuous. Since $\int_{-\infty}^{\infty} g(s)\,\mathrm{d}s \ne 0$, $f(t) = 0$. □

Example 1.6. Define $\delta : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ by $\delta(f) := f(0)$ for any $f \in \mathcal{S}(\mathbb{R})$. Then δ is a tempered distribution.

Proof. The linearity of δ is straightforward. The continuity of δ follows from the inequality shown as follows:

$$|\delta(f_1) - \delta(f_2)| = |f_1(0) - f_2(0)| \le |f_1 - f_2|_{0,0},$$

for any $f_1, f_2 \in \mathcal{S}(\mathbb{R})$. □

2 Operations in $\mathcal{S}'(\mathbb{R})$

2.1 Multiplication with a Function

Definition (Multiplication with a function)**.** *Given any $g \in PBC^{\infty}(\mathbb{R})$ and any $\phi \in \mathcal{S}'(\mathbb{R})$, $g\phi : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is defined by*

$$g\phi(f) := \phi(fg),$$

for every $f \in \mathcal{S}(\mathbb{R})$.

Recall that we have seen in Proposition 1.1 that $fg \in \mathcal{S}(\mathbb{R})$ when $f \in \mathcal{S}(\mathbb{R})$ and $g \in PBC^\infty(\mathbb{R})$, so the expression $\phi(fg)$ in this definition makes sense. By the product rule, if $\lim_{n\to+\infty} |f_n - f|_{k,l} = 0$ for f_n's and f in $\mathcal{S}(\mathbb{R})$ and any $k, l \in \mathbb{N}$, $\lim_{n\to+\infty} |f_n g - fg|_{k,l} = 0$ always holds for $g \in PBC^\infty(\mathbb{R})$. Hence, $g\phi$ is continuous as well, so $g\phi \in \mathcal{S}'(\mathbb{R})$.

Proposition 2.1. *For any $f, g \in PBC^\infty(\mathbb{R})$ and $\phi \in \mathcal{S}'(\mathbb{R})$, $(fg)\phi = f(g\phi)$.*

Proof. For any $u \in \mathcal{S}(\mathbb{R})$, $f(g\phi)(u) = g\phi(fu) = \phi(gfu) = ((fg)\phi)(u)$, showing our claim. □

2.2 Weak Derivative

Definition (Weak derivatives)**.** *For any $\phi \in \mathcal{S}'(\mathbb{R})$, its* weak derivative *$\phi'$ is a linear functional on $\mathcal{S}(\mathbb{R})$ given by*

$$\phi'(f) := -\phi(f')$$

for any $f \in \mathcal{S}(\mathbb{R})$.

Observe that $|f'|_{k,l} = |f|_{k,l+1}$ for any $k, l \in \mathbb{N}$, so one can easily deduce that $\phi' \in \mathcal{S}'(\mathbb{R})$ provided that $\phi \in \mathcal{S}'(\mathbb{R})$.

The weak derivative coincide with the derivative for every bounded smooth function. (Boundedness is to guarantee that this function does define a tempered distribution here.)

Proposition 2.2. *Suppose that $f : \mathbb{R} \to \mathbb{C}$ is bounded, continuous and piecewise C^1. Then, $(f^*)' = (f')^*$.*

Proof. For any $g \in \mathcal{S}(\mathbb{R})$, as f is bounded, applying integration by parts, one obtains

$$(f^*)'(g) = -f^*(g') = -\int_{-\infty}^{\infty} f(x)g'(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} f'(x)g(x)\,\mathrm{d}x = (f')^*(g).$$

Therefore, $(f^*)' = (f')^*$. □

2.3 Fourier Transform

Definition (Fourier transform of a Schwartz function)**.** *For any $f \in \mathcal{S}(\mathbb{R})$, its* Fourier transform *$\mathcal{F}\{f\} : \mathbb{R} \to \mathbb{R}$ is defined by*

$$\mathcal{F}\{f\}(\omega) := \int_{-\infty}^{\infty} f(t)\exp(-it\omega)\,\mathrm{d}t,$$

for any $\omega \in \mathbb{R}$.

This integral makes sense because, as we showed before, Schwartz functions are all absolutely integrable.

Proposition 2.3. *The Fourier transform maps $\mathcal{S}(\mathbb{R})$ to itself. Moreover, the Fourier transform is also continuous in the sense that for f_n's and f in $\mathcal{S}(\mathbb{R})$, given that*

$$\lim_{n\to+\infty} |f_n - f|_{k,l} = 0$$

for any $k, l \in \mathbb{N}$, we always have

$$\lim_{n\to+\infty} |\mathcal{F}\{f_n\} - \mathcal{F}\{f\}|_{k,l} = 0$$

for any $k, l \in \mathbb{N}$.

Proof. For any $k, l \in \mathbb{N}$,

$$\begin{aligned}
|\mathcal{F}\{f\}|_{k,l} = \sup_{\omega\in\mathbb{R}} \left|\omega^k \mathcal{F}\{f\}^{(l)}(\omega)\right| &= \sup_{\omega\in\mathbb{R}} \left|\omega^k \int_{-\infty}^{\infty} t^l f(t) \exp(-it\omega)\,\mathrm{d}t\right| \\
&= \sup_{\omega\in\mathbb{R}} \left|\int_{-\infty}^{\infty} t^l f(t) \frac{\partial^k}{\partial t^k} \exp(-it\omega)\,\mathrm{d}t\right| \\
&= \sup_{\omega\in\mathbb{R}} \left|\int_{-\infty}^{\infty} \exp(-it\omega) \frac{\mathrm{d}^k}{\mathrm{d}t^k} t^l f(t)\,\mathrm{d}t\right| \\
&\le \int_{-\infty}^{\infty} \left|\frac{\mathrm{d}^k}{\mathrm{d}t^k} t^l f(t)\right| \mathrm{d}t \\
&= \int_{-\infty}^{\infty} \left|(1+t^2)\frac{\mathrm{d}^k}{\mathrm{d}t^k} t^l f(t)\right| \frac{1}{1+t^2}\,\mathrm{d}t \\
&\le \sup_{t\in\mathbb{R}} \left|(1+t^2)\frac{\mathrm{d}^k}{\mathrm{d}t^k} t^l f(t)\right|.
\end{aligned}$$

Now notice that by the product rule, there exist integers $a_{k',l'}$ for $0 \le k' \le k$ and $0 \le l' \in l+2$ such that

$$(1+t^2)\frac{\mathrm{d}^k}{\mathrm{d}t^k} t^l f(t) = \sum_{\substack{0\le k'\le k \\ 0\le l'\le l+2}} a_{k',l'} t^{l'} f^{(k')}(t)$$

for any t. Hence,

$$|\mathcal{F}\{f\}|_{k,l} \le \sup_{t\in\mathbb{R}} \left|(1+t^2)\frac{\mathrm{d}^k}{\mathrm{d}t^k} t^l f(t)\right| \le \sum_{\substack{0\le k'\le k \\ 0\le l'\le l+2}} |a_{k',l'}| \cdot |f|_{l',k'}.$$

The last expression is always finite, so $\mathcal{F}\{f\}$ is Schwartz provided that f is Schwartz. In addition, it is apparent that the continuity also follows from this inequality. ☐

Proposition 2.4. *For any $f, g \in \mathcal{S}(\mathbb{R})$,*

$$\int_{-\infty}^{\infty} \mathcal{F}\{f\}(t) g(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} f(t)\mathcal{F}\{g\}(t)\,\mathrm{d}t.$$

Proof. It is an immediate consequence of the Fubini's theorem. □

Proposition 2.5. *For any $f \in \mathcal{S}(\mathbb{R})$ and any $t \in \mathbb{R}$, $\mathcal{F}^2\{f\}(t) = 2\pi f(-t)$.*

Proof. Fix an arbitrary $g \in \mathcal{S}(\mathbb{R})$. On the one hand, note that

$$\mathcal{F}\{g\}(0) = \int_{-\infty}^{\infty} g(s)\,\mathrm{d}s, \tag{1}$$

so Lemma 1.4 implies that

$$\lim_{n\to+\infty} n \int_{-\infty}^{\infty} \mathcal{F}^2\{f\}(s) g(n(t-s))\,\mathrm{d}s = \mathcal{F}^2\{f\}(t)\mathcal{F}\{g\}(0), \tag{2}$$

for any $t \in \mathbb{R}$.

On the other hand, by change of variables and Fubini's theorem,

$$\begin{aligned}
& n \int_{-\infty}^{\infty} \mathcal{F}^2\{f\}(s) g(n(t-s))\,\mathrm{d}s \\
&= n \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{F}\{f\}(\omega)\exp(-i\omega s) g(n(t-s))\,\mathrm{d}\omega\mathrm{d}s \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{F}\{f\}(\omega)\exp(-i\omega(t-s/n)) g(s)\,\mathrm{d}s\mathrm{d}\omega \\
&= \int_{-\infty}^{\infty} \mathcal{F}\{f\}(\omega)\exp(-i\omega t)\mathcal{F}\{g\}(-\omega/n)\,\mathrm{d}\omega \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s)\exp(-i\omega s)\exp(-i\omega t)\mathcal{F}\{g\}(-\omega/n)\,\mathrm{d}s\mathrm{d}\omega \\
&= n \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s)\exp(in\omega(s+t))\mathcal{F}\{g\}(\omega)\,\mathrm{d}\omega\mathrm{d}s \\
&= n \int_{-\infty}^{\infty} f(s)\mathcal{F}^2\{g\}(-n(s+t))\,\mathrm{d}s.
\end{aligned}$$

Taking the limit when n approaches infinity, one obtains from Lemma 1.4 and (1) that

$$\lim_{n\to+\infty} n \int_{-\infty}^{\infty} \mathcal{F}^2\{f\}(s) g(n(t-s))\,\mathrm{d}s = f(-t)\mathcal{F}^3\{g\}(0). \tag{3}$$

Combining (2) and (3), we have $\mathcal{F}^2\{f\}(t)\mathcal{F}\{g\}(0) = f(-t)\mathcal{F}^3\{g\}(0)$ for any two Schwartz functions f and g. Now we take $g : t \mapsto \exp(-t^2/2)$. Hence, since $z \mapsto \exp(-z^2/2)$ is an entire function on $\mathbb{C}$, a standard complex-analytic argument yields

$$\begin{aligned}
\mathcal{F}\{g\}(\omega) &= \int_{-\infty}^{\infty} \exp(-t^2/2 - it\omega)\,\mathrm{d}t \\
&= \exp(-\omega^2/2)\int_{-\infty}^{\infty} \exp(-(t+i\omega)^2/2)\,\mathrm{d}t \\
&= \exp(-\omega^2/2)\int_{-\infty}^{\infty} \exp(-t^2/2)\,\mathrm{d}t = \sqrt{2\pi}\exp(-\omega^2/2) = \sqrt{2\pi} g(\omega).
\end{aligned}$$

In short, $\mathcal{F}\{g\} = \sqrt{2\pi}g$, implying that $\mathcal{F}^3\{g\}(0) = 2\pi\mathcal{F}\{g\}(0)$. Thus, we can conclude that $\mathcal{F}^2\{f\}(t) = 2\pi f(-t)$ for any $f \in \mathcal{S}(\mathbb{R})$ and $t \in \mathbb{R}$. □

An immediate consequence is that applying Fourier transform to a Schwartz function for four times is simply multiplying it by $4\pi^2$. Therefore, the inverse of Fourier transform exists and we shall call it the inverse Fourier transform.

Corollary 2.6. *The inverse Fourier transform $\mathcal{F}^{-1}\{f\}$ of a Schwartz function f is given by*

$$\mathcal{F}^{-1}\{f\}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\exp(it\omega)\,\mathrm{d}t$$

for any $\omega \in \mathbb{R}$.

In light of Proposition 2.4, we can define the Fourier transform of a tempered distribution rather than a Schwartz function.

Definition (Fourier transform of a tempered distribution)**.** *For any $\phi \in \mathcal{S}'(\mathbb{R})$, its* Fourier transform *$\mathcal{F}\{\phi\} \in \mathcal{S}'$ is defined by*

$$\mathcal{F}\{\phi\}(f) \coloneqq \phi(\mathcal{F}\{f\}),$$

for any $f \in \mathcal{S}(\mathbb{R})$.

Proposition 2.3 shows the soundness of this definition.

Proposition 2.7. *The Fourier transform defined on $\mathcal{S}'(\mathbb{R})$ is bijective.*

Proof. Suppose that $\mathcal{F}\{\phi\} = \mathcal{F}\{\psi\}$ for $\phi, \psi \in \mathcal{S}'(\mathbb{R})$. Then for any $f \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}\phi(f) = \phi(\mathcal{F}\{\mathcal{F}^{-1}\{f\}\}) &= \mathcal{F}\{\phi\}(\mathcal{F}^{-1}\{f\}) \\ &= \mathcal{F}\{\psi\}(\mathcal{F}^{-1}\{f\}) = \psi(\mathcal{F}\{\mathcal{F}^{-1}\{f\}\}) = \psi(f),\end{aligned}$$

showing that Fourier transform on $\mathcal{S}'(\mathbb{R})$ is injective.

For any $\phi \in \mathcal{S}'(\mathbb{R})$, we claim that the Fourier transform of $(1/4\pi^2)\mathcal{F}^3\{\phi\} \in \mathcal{S}'(\mathbb{R})$ is ϕ, from which the surjectivity of Fourier transform follows immediately. This claim can be shown by using Proposition 2.5 in the following manner. For every $f \in \mathcal{S}(\mathbb{R})$,

$$\mathcal{F}\{(1/4\pi^2)\mathcal{F}^3\{\phi\}\}(f) = \phi\left(\frac{1}{4\pi^2}\mathcal{F}^4\{f\}\right) = \phi(f),$$

indicating that the Fourier transform of $(1/4\pi^2)\mathcal{F}^3\{\phi\} \in \mathcal{S}'(\mathbb{R})$ is indeed ϕ. □

Example 2.8. We calculate the Fourier transform of the following tempered distributions.

1. Let $f : \mathbb{R} \to \mathbb{C}$ be an absolutely integrable piecewise continuous function. As in Example 1.3, $f^* : g \mapsto \int_{-\infty}^{\infty} f(t)g(t)\,\mathrm{d}t$ is a tempered distribution. For any $g \in \mathcal{S}(\mathbb{R})$, from Fubini's theorem, we have

$$\begin{aligned}\mathcal{F}\{f^*\}(g) = f^*(\mathcal{F}\{g\}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\omega)g(t)\exp(-it\omega)\,\mathrm{d}t\mathrm{d}\omega \\ &= \int_{-\infty}^{\infty} g(t)\left(\int_{-\infty}^{\infty} f(\omega)\exp(-it\omega)\,\mathrm{d}\omega\right)\mathrm{d}t.\end{aligned}$$

   Therefore, if we extend the definition of Fourier transform of Schwartz functions to absolutely integrable functions by simply letting

$$\mathcal{F}\{f\}(t) \coloneqq \int_{-\infty}^{\infty} f(\omega)\exp(-it\omega)\,\mathrm{d}\omega$$

   for any absolutely integrable function f and real number t, then we have $\mathcal{F}\{f^*\} = \mathcal{F}\{f\}^*$.

2. $\mathcal{F}\{\delta\}(g) = \delta(\mathcal{F}\{g\}) = \mathcal{F}\{g\}(0) = \int_{-\infty}^{\infty} g(t)\,\mathrm{d}t$ for any Schwartz function g, so $\mathcal{F}\{\delta\} = 1^*$, where 1 denotes the function constantly equal to 1.

2.4 Convolution

Definition. *Assume that $f : \mathbb{R} \to \mathbb{C}$ is a bounded piecewise continuous function satisfying that there exists some $R > 0$ such that $f(t) = 0$ for any $t > R$. Then, for any $\phi \in \mathcal{S}'(\mathbb{R})$, define the* convolution *$f * \phi$ of f and ϕ as $\mathcal{F}^{-1}\{\mathcal{F}\{f\}\mathcal{F}\{\phi\}\} \in \mathcal{S}'(\mathbb{R})$.*

Remark 2.9. *In Example 2.8, we gave the expression of the Fourier transform $\mathcal{F}\{f\}$ of the function f described above. For any choice of $R > 0$ which satisfies that $f(t) = 0$ for any $t > R$, we have*

$$\mathcal{F}\{f\}(t) = \int_{-R}^{R} f(\omega)\exp(-i\omega t)\,\mathrm{d}\omega.$$

Notice that this function is of C^∞ class with its l-th derivative being

$$\mathcal{F}\{f\}^{(l)}(t) = (-i)^l \int_{-R}^{R} \omega^l f(\omega)\exp(-i\omega t)\,\mathrm{d}\omega.$$

Hence, for any $l \in \mathbb{N}$,

$$|\mathcal{F}\{f\}|_{0,l} \le \int_{-R}^{R} |\omega^l f(\omega)|\,\mathrm{d}\omega \le \frac{2R^{l+1}}{l+1} \cdot \sup_{\omega\in\mathbb{R}} |f(\omega)| < \infty.$$

This shows that $\mathcal{F}\{f\} \in BC^\infty(\mathbb{R})$, so the multiplication in the definition of the convolution makes sense.

Proposition 2.10. *Suppose that $f : \mathbb{R} \to \mathbb{C}$ is a bounded piecewise continuous function satisfying that there exists some $R > 0$ such that $f(t) = 0$ for any $t > R$, and $g : \mathbb{R} \to \mathbb{C}$ is an absolutely integrable function. Then, $f * (g^*) = (f * g)^*$, where $f * g : \mathbb{R} \to \mathbb{C}$ is given by*

$$(f * g)(t) := \int_{-\infty}^{\infty} f(s)g(t-s)\,\mathrm{d}s = \int_{-\infty}^{\infty} f(t-s)g(s)\,\mathrm{d}s$$

for any $t \in \mathbb{R}$.

Proof. Since f is bounded and g is absolutely integrable, the function $f * g$ is well-defined. As f is absolutely integrable as well, by Fubini's theorem,

$$\begin{aligned}\int_{-\infty}^{\infty} |(f * g)(t)|\,\mathrm{d}t &\leq \int_{-\infty}^{\infty} |f(s)| \left(\int_{-\infty}^{\infty} |g(t-s)|\,\mathrm{d}t \right) \mathrm{d}s \\ &\leq \int_{-\infty}^{\infty} |f(s)|\,\mathrm{d}s \int_{-\infty}^{\infty} |g(t)|\,\mathrm{d}t < \infty.\end{aligned}$$

Therefore, $f * g$ is absolutely integrable, so it *does* give a tempered distribution $(f * g)^*$.

By definition, for any real number ω,

$$\begin{aligned}\mathcal{F}\{f * g\}(\omega) &= \int_{-\infty}^{\infty} \exp(-it\omega) \left(\int_{-\infty}^{\infty} f(s)g(t-s)\,\mathrm{d}s \right) \mathrm{d}t \\ &= \int_{-\infty}^{\infty} \exp(-is\omega) f(s) \left(\int_{-\infty}^{\infty} \exp(-i(t-s)\omega) g(t-s)\,\mathrm{d}t \right) \mathrm{d}s \\ &= \int_{-\infty}^{\infty} \exp(-is\omega) f(s) \mathcal{F}\{g\}(\omega)\,\mathrm{d}s = \mathcal{F}\{f\}(\omega)\mathcal{F}\{g\}(\omega).\end{aligned}$$

Hence, $\mathcal{F}\{f * g\} = \mathcal{F}\{f\}\mathcal{F}\{g\}$. Using the properties that we have previously proved, we have

$$\begin{aligned}\mathcal{F}\{f * (g^*)\} = \mathcal{F}\{f\}\mathcal{F}\{g^*\} &= \mathcal{F}\{f\}(\mathcal{F}\{g\}^*) \\ &= (\mathcal{F}\{f\}\mathcal{F}\{g\})^* = \mathcal{F}\{f * g\}^* = \mathcal{F}\{(f * g)^*\}.\end{aligned}$$

Consequently, by the bijectivity of Fourier transform shown in Proposition 2.7, we have $f * (g^*) = (f * g)^*$. □

Proposition 2.11. *Let f be a function satisfying the conditions described in the definition of convolution. Then, for any $\phi \in \mathcal{S}'(\mathbb{R})$, $(f * \phi)' = f * \phi'$.*

Proof. Throughout this proof, we set p to be the polynomial given by $p(t) := t$ for any real number t. By definition, $p \in BC^{\infty}(\mathbb{R})$, so we can always multiply a tempered distribution by p.

For any $u \in \mathcal{S}(\mathbb{R})$ and $\omega \in \mathbb{R}$,

$$\mathcal{F}\{u\}'(\omega) = -i \int_{-\infty}^{\infty} tu(t) \exp(-it\omega)\,\mathrm{d}t = -i\mathcal{F}\{pu\}(\omega),$$

showing that $\mathcal{F}\{u\}' = -i\mathcal{F}\{pu\}$. Hence, for any $\psi \in \mathcal{S}'(\mathbb{R})$,

$$\mathcal{F}\{\psi'\}(u) = -\psi(\mathcal{F}\{u\}') = i\psi(\mathcal{F}\{pu\}) = (ip\mathcal{F}\{\psi\})(u).$$

As u is an arbitrary Schwartz function, we deduce that $\mathcal{F}\{\psi'\} = ip\mathcal{F}\{\psi\}$.

Now for the given f and ψ, we have

$$\mathcal{F}\{(f * \phi)'\} = ip\mathcal{F}\{f * \phi\} = \mathcal{F}\{f\}(ip\mathcal{F}\{\phi\}) = \mathcal{F}\{f\}\mathcal{F}\{\phi'\} = \mathcal{F}\{f * \phi'\}.$$

As the Fourier transform is bijective, we conclude that $(f * \phi)' = f * \phi'$. $\square$

3 Verification of Derivations in Guan's Report

The differential equation solved in Method 3 of Guan's report is

$$\dot{x} + Px = q, \tag{4}$$

where $P \in \mathbb{R}$ is a constant and $q : \mathbb{R} \to \mathbb{R}$ is a function. The initial condition is $x(0) = x_0 \in \mathbb{R}$. For simplicity, we will assume that q is continuous.

In this section, we shall make use of the tools in the previous sections to see why the Green function G, which is to appear later, gives a solution to the differential equation (4). The solution x given by G, which will be defined at the end of this section, does not necessarily satisfy the initial condition, but as Guan presented a way to handle this problem, we shall forget about the initial condition in what follows.

The first step is to find a tempered distribution ϕ such that

$$\phi' + P\phi = \delta. \tag{5}$$

When $P = 0$, we only need to find one $\phi \in \mathcal{S}'(\mathbb{R})$ such that $\phi' = \delta$, meaning that for any $f \in \mathcal{S}(\mathbb{R})$,

$$\delta(f) = f(0) = -\int_{-\infty}^{\infty} H(t)f'(t)\,\mathrm{d}t = -H^*(f') = (H^*)'(f),$$

where $H(t)$ equals 0 for negative t and equals 1 for non-negative t. Therefore, ϕ can be taken to be H^* when $P = 0$.

Now we consider the case where $P \neq 0$. Let $\hat{G} \in BC^\infty(\mathbb{R}) \subseteq PBC^\infty(\mathbb{R})$ be given by $\hat{G}(\omega) := 1/(i\omega + P)$ for any $\omega \in \mathbb{R}$. Notice that the assumption that $P \neq 0$ guarantees $\hat{G} \in BC^\infty(\mathbb{R})$. Applying Fourier transform on both sides of (5) yields

$$\frac{1}{\hat{G}}\mathcal{F}\{\phi\} = 1^*.$$

Therefore, $\mathcal{F}\{\phi\} = \hat{G} \cdot 1^* = \hat{G}^*$, indicating that $\phi = \mathcal{F}^{-1}\{\hat{G}^*\}$.

As $\hat{G}$ is a function, it is a reasonable guess that $\phi = \mathcal{F}^{-1}\{\hat{G}\}^*$, where

$$\mathcal{F}^{-1}\{\hat{G}\} : t \mapsto \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{G}(\omega)\exp(it\omega)\,\mathrm{d}\omega. \tag{6}$$

Nevertheless, the integrand here is not absolutely integrable, meaning that we cannot directly apply Corollary 2.6. To derive an expression of $\mathcal{F}^{-1}\{\hat{G}\}$, we shall make use of several facts from Guan's report.

Fact 3.1 (Facts proved in Guan's report)**.** *For $R > 0$, set $g_R : \mathbb{R} \to \mathbb{C}$ as*

$$g_R : t \mapsto \frac{1}{2\pi} \int_{-R}^{R} \hat{G}(\omega) \exp(it\omega) \, \mathrm{d}\omega. \tag{7}$$

Moreover, define $g_\infty : \mathbb{R} \to \mathbb{C}$ by letting for any t,

$$g_\infty(t) := \begin{cases} \exp(-Pt) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

when $P > 0$, and

$$g_\infty(t) := \begin{cases} 0 & \text{if } t > 0 \\ \exp(-Pt) & \text{if } t \leq 0 \end{cases}$$

when $P < 0$. Then, for any $R > |P|$ and $t \neq 0$,

$$|g_R(t) - g_\infty(t)| \leq \frac{\pi(1 - \exp(-R|t|))}{(R - |P|)|t|}. \tag{8}$$

In particular, Fact 3.1 implies that g_R converges pointwise to g_∞ (possibly except at 0) as R approaches infinity.

Proposition 3.2. *When $P \neq 0$, $\phi := g_\infty^*$ is the unique tempered distribution satisfying (5).*

The following proof only makes use of the estimate (8), and does not use the explicit expressions of g_∞ in Fact 3.1.

Proof. Note that g_∞ is absolutely integrable, so g_∞^* *is* a tempered distribution. The uniqueness follows from the fact that Fourier transform is invertible. Hence, it only suffices to show that $\mathcal{F}^{-1}\{\hat{G}^*\} = g_\infty^*$.

Consider the sequence $(g_n)_{n=1}^\infty$. For any $f \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}
\mathcal{F}^{-1}\{\hat{G}^*\}(f) &= \mathcal{F}^{-1}\{\hat{G}^*\}(\mathcal{F}\{\mathcal{F}^{-1}\{f\}\}) \\
&= \hat{G}^*(\mathcal{F}^{-1}\{f\}) \\
&= \int_{-\infty}^{\infty} \hat{G}(\omega)\mathcal{F}^{-1}\{f\}(\omega) \, \mathrm{d}\omega \\
&\overset{(a)}{=} \frac{1}{2\pi} \lim_{n\to+\infty} \int_{-n}^{n} \hat{G}(\omega) \left(\int_{-\infty}^{\infty} f(t) \exp(it\omega) \, \mathrm{d}t \right) \mathrm{d}\omega \\
&\overset{(b)}{=} \frac{1}{2\pi} \lim_{n\to+\infty} \int_{-\infty}^{\infty} f(t) \left(\int_{-n}^{n} \hat{G}(\omega) \exp(it\omega) \, \mathrm{d}\omega \right) \mathrm{d}t \\
&= \lim_{n\to+\infty} \int_{-\infty}^{\infty} f(t) g_n(t) \, \mathrm{d}t \\
&\overset{(c)}{=} \int_{-\infty}^{\infty} f(t) g_\infty(t) \, \mathrm{d}t.
\end{aligned}$$

Here, (b) is due to Fubini's theorem, and both (a) and (c) follow from dominated convergence theorem. One dominant for (a) can be simply $|\hat{G}\mathcal{F}^{-1}\{f\}|$. To find a dominant for (c), use Fact 3.1, and we have for any $n \geq |P|+1$,

$$\begin{aligned}
\sup_{t\neq 0}|g_n(t)| &\leq 1+\frac{\pi}{n-|P|}\sup_{t\neq 0}\frac{1-\exp(-n|t|)}{|t|} \\
&= 1+\frac{\pi}{n-|P|}\sup_{t>0}\frac{1-\exp(-nt)}{t} \\
&\leq 1+\frac{\pi}{n-|P|}\sup_{t>0} n\exp(-nt) \\
&= 1+\frac{n\pi}{n-|P|} = 1+\pi\left(1+\frac{|P|}{n-|P|}\right) \leq 1+\pi(1+|P|).
\end{aligned}$$

Here the second inequality follows from the mean value theorem. Hence, one choice of dominant for (c) can be $(1+\pi+|P|\pi)|f|$.

Now that we have shown that

$$\mathcal{F}^{-1}\{\hat{G}^*\}(f) = \int_{-\infty}^{\infty} f(t)g_\infty(t)\,\mathrm{d}t,$$

we can conclude that the unique tempered distribution $\phi = \mathcal{F}^{-1}\{\hat{G}^*\}$ satisfying (5) is precisely g_∞^*. □

Now we consider the differential equation (4). For the continuous function q appearing in (4) and an arbitrary $T>0$, define $q_T : \mathbb{R} \to \mathbb{R}$ by

$$q_T(t) \coloneqq \begin{cases} q(t) & \text{if } 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

for any $t \in \mathbb{R}$. Then q_T satisfies the conditions of the function in the definition of convolution, so we can take the convolution of q_T and an arbitrary tempered distribution.

Now that we have got a tempered distribution ϕ satisfying (5) for any fixed P, applying Proposition 2.11, we have

$$(q_T * \phi)' + P(q_T * \phi) = q_T * \phi' + Pq_T * \phi = q_T * \delta - Pq_T * \phi + Pq_T * \phi = q_T^*.$$

Define $G : \mathbb{R}^2 \to \mathbb{R}$ by, for any $t, \xi \in \mathbb{R}$,

$$G(t,\xi) \coloneqq g_\infty(t-\xi)$$

when $P \neq 0$ and

$$G(t,\xi) \coloneqq H(t-\xi)$$

when $P = 0$. Set

$$x_T(t) \coloneqq \int_{-\infty}^{\infty} G(t,\xi)q_T(\xi)\,\mathrm{d}\xi = \int_0^T G(t,\xi)q(\xi)\,\mathrm{d}\xi$$

for each real number t. Thus, by Proposition 2.10, $q_T * \phi = x_T^*$. Furthermore, the expressions of g_∞ and H gives that

1. when $P < 0$,

$$x_T(t) = \int_{\max\{0,t\}}^{T} \exp(-P(t-\xi))q(\xi)\,\mathrm{d}\xi,$$

   for $t < T$ and $x_T(t) = 0$ for $t \geq T$;

2. when $P = 0$,

$$x_T(t) = \int_{0}^{\min\{t,T\}} q(\xi)\,\mathrm{d}\xi,$$

   for $t > 0$ and $x_T(t) = 0$ for $t \leq 0$;

3. when $P > 0$,

$$x_T(t) = \int_{0}^{\min\{t,T\}} \exp(-P(t-\xi))q(\xi)\,\mathrm{d}\xi,$$

   for $t > 0$ and $x_T(t) = 0$ for $t \leq 0$.

Hence, x_T is bounded, continuous and piecewise C^1, for the fixed $T \in (0, +\infty)$. By Proposition 2.2, we then have $(x_T^*)' = (x_T')^*$, and as a consequence,

$$(x_T' + Px_T - q_T)^* = 0^*.$$

As $x_T' + Px_T - q_T$ is continuous on $(0, T)$, by Proposition 1.5,

$$x_T'(t) + Px_T(t) = q_T(t) = q(t)$$

for any $t \in (0, T)$.

References

[1] E. M. Stein and R. Shakarchi. *Fourier analysis: an introduction.* Princeton University Press, 2011.