Reference book [Teschl]

Problem 5.14 Yui Hatanaka

$$v(x) = e^{\frac{1}{2}\int^x g(t)dt} u(x)$$

$$\dot{v}(x) = \left(\frac{1}{2}\int^x g(t)dt\right)' e^{\frac{1}{2}\int^x g(t)dt} u(x) + e^{\frac{1}{2}\int^x g(t)dt} \dot{u}(x)$$

$$= \frac{1}{2}g(x) e^{\frac{1}{2}\int^x g(t)dt} u(x) + e^{\frac{1}{2}\int^x g(t)dt} \dot{u}(x)$$

$$\ddot{v}(x) = \frac{1}{2}g(x)\left\{\frac{1}{2}g(x) e^{\frac{1}{2}\int^x g(t)dt} u(x) + e^{\frac{1}{2}\int^x g(t)dt} \dot{u}(x)\right\}$$

$$+ \frac{1}{2}\dot{g}(x) e^{\frac{1}{2}\int^x g(t)dt} u(x) + \frac{1}{2}g(x) e^{\frac{1}{2}\int^x g(t)dt} \dot{u}(x)$$

$$+ e^{\frac{1}{2}\int^x g(t)dt} \ddot{u}(x)$$

$$= \frac{1}{4}g(x)^2 e^{\frac{1}{2}\int^x g(t)dt} u(x) + \frac{1}{2}g(x) e^{\frac{1}{2}\int^x g(t)dt} \dot{u}(x)$$

$$+ \frac{1}{2}\dot{g}(x) e^{\frac{1}{2}\int^x g(t)dt} u(x) + \frac{1}{2}g(x) e^{\frac{1}{2}\int^x g(t)dt} \dot{u}(x)$$

$$+ e^{\frac{1}{2}\int^x g(t)dt} \ddot{u}(x)$$

$$= e^{\frac{1}{2}\int^x g(t)dt}\Big(\frac{1}{4}g(x)^2 u(x) + g(x)\dot{u}(x)$$

$$+ \frac{1}{2}\dot{g}(x)u(x) + \ddot{u}(x)\Big)$$

$$\ddot{v}(x) + \left(f(x) - \frac{1}{2}\dot{g}(x) - \frac{1}{4}g(x)^2\right)v(x)$$

$$= e^{\frac{1}{2}\int^x g(t)dt}\left(\frac{1}{4}g(x)^2 u(x) + g(x)\dot{u}(x) + \frac{1}{2}\dot{g}(x)u(x) + \ddot{u}(x)\right)$$

$$+ \left(f(x) - \frac{1}{2}\dot{g}(x) - \frac{1}{4}g(x)^2\right) e^{\frac{1}{2}g(t)dt} u(x)$$

$$= e^{\frac{1}{2}\int^x g(t)dt}\Big(\frac{1}{4}g(x)^2 u(x) + g(x)\dot{u}(x) + \frac{1}{2}\dot{g}(x)u(x) + \ddot{u}(x)$$

$$+ f(x)u(x) - \frac{1}{2}\dot{g}(x)u(x) - \frac{1}{4}g(x)^2 u(x)\Big)$$

$$= e^{\frac{1}{2}\int^x g(t)dt}\left(g(x)\dot{u}(x) + f(x)u(x) + \ddot{u}(x)\right)$$

then $h(x) = g(x)\dot{u}(x) + f(x)u(x) + \ddot{u}(x)$

So, $\ddot{v}(x) + \left(f(x) - \frac{1}{2}\dot{g}(x) - \frac{1}{4}g(x)^2\right)v(x)$

$$= e^{\frac{1}{2}\int^x g(t)dt} h(x)$$