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$\omega_\pm(x)$ is a closed invariant set.

No.

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Def:

The $w_\pm$-limit set of the orbit $\gamma(x)$ is the set of those points $y \in M$ for which there exists a sequence $t_n \to \pm\infty$ with $\phi(t_n, x) \to y$.

Theorem:

The set $w_\pm(x)$ is a closed invariant set.

Proof:

① closed:

We only need to prove that if a point y is in the closure of $w_\pm(x)$, then $y \in w_\pm(x)$.

Suppose y is in the closure of $w_\pm(x)$. Then, by the definition of closure, we can always find a sequence y_n $(\in w_\pm(x))$ which converges to y. Then, $\forall \varepsilon > 0$, $\exists N_1$ s.t. $|y - y_n| < \frac{\varepsilon}{2}$, $n > N_1$. Since y_n is in $w_\pm(x)$, there exists a sequence $t_n \to \infty$, $\forall \varepsilon > 0$, $\exists N_2$ s.t. $|\phi(t_n, x) - y_n| < \frac{\varepsilon}{2}$, $n > N_2$. Then, we can conclude that $\forall \varepsilon > 0$, $\exists N_1, N_2$ s.t. $|\phi(t_n, x) - y| < \varepsilon$, $n > \max\{N_1, N_2\}$. By definition, this means that $y \in w_\pm(x)$. $\blacksquare$

② invariant:

We need to prove that if $y \in w_\pm(x)$, then $\phi(t, y) \in w_\pm(x)$.

In fact, $\phi(t, y) = \phi(t, \lim_{n\to\infty} \phi(t_n, x)) = \phi(t, \phi(\lim_{n\to\infty} t_n, x)) = \phi(\lim_{n\to\infty} t_n + t, x)$. Since $\lim_{n\to\infty} t_n = \infty$ and $\infty + t = \infty$, then there is a new sequence $t_m = t_n + t$ and $t_m \to \infty$, so $\phi(t, y) = \phi(\lim_{m\to\infty} t_m, x) \in w_\pm(x)$ by definition of $w_\pm(x)$.

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